

What percentile sound levels identify: *energy bounds, background recovery, and source combination*

Omri Abas

Independent researcher, Kadima-Zoran, Israel

omri.abas1@gmail.com

ORCID: 0009-0003-4857-0790

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Abstract

We determine what percentile sound levels identify about equivalent energy, background recovery and source combination. Under a nonnegative additive-energy model, partial identification yields sharp bounds over compatible distributions. Finite percentile reports leave equivalent levels unbounded above; support bounds yield sharp intervals with endpoint conditions. Background and source mean-energy intervals follow from known total distributions or percentile packages, with declared activity, support and mean constraints. Recovery conditions and an error identity separate source-off selection error from background change. Without support restrictions, substituting total-record L_{95} for background L_{eq} has no finite error bound in either direction under at least 5% exact inactivity; at least 95% gives a sharp, unattained overestimation supremum of approximately 0.4576 dB, while underestimation remains unbounded. Component percentiles yield a sharp lower bound under arbitrary dependence and a sharp envelope under justified mutual independence. Additional information gives exact independent-source intervals in specified regimes, finite reductions and certified grid bounds on bounded supports. Two fixed finite marginals permit probability, rank and exact transport bounds under dependence budgets. Percentile subtraction permits unlimited overestimation; pooling identifies the closed interval between extreme window percentiles. These results inform assessment standards and measurement reporting while distinguishing identification within a measured window from representativeness beyond it.

Keywords: percentile sound levels; equivalent continuous sound pressure level; background and residual sound; partial identification; sharp quantile bounds; dependence and quantile aggregation; ISO 1996-2

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1. Introduction

An environmental-noise measurement usually contains both the source under investigation and residual sound. Recovering the source equivalent level requires the background mean energy over the assessed period. Can a low percentile such as L_{90} or L_{95} supply it? More generally, what do the total record, its

equivalent level and its reported percentiles establish about the background, and what additional evidence is needed?

An equivalent level summarizes mean sound energy; a percentile locates a rank in the measured distribution. Both are useful and both are expressed in decibels, but their common unit does not make them interchangeable. The issue is what a descriptor determines when converted, subtracted, added or pooled with other measurements.

That question arises directly in environmental-noise practice. ISO 1996-2:2017 proposes a low percentile of the total record as a possible representation of residual sound in Annex I.2.1, while Annex I.2.2 estimates the equivalent level of the residual sound from L_{50} and L_{90} or L_{95} through Gaussian formulas [1]. Clause 10.4 of the same document performs residual correction by subtracting equivalent mean energies, an operation of a different mathematical kind. BS 4142:2014+A1:2019 likewise keeps two roles apart: ambient and residual sound are expressed as equivalent levels for source correction, whereas the background sound level is an $L_{A90,T}$ used in the rating-to-background comparison [2]. Other procedures combine percentile-labelled quantities more directly. The 2026 United Kingdom wind-turbine guidance averages $L_{A90,10min}$ readings within wind-speed bins and then applies logarithmic subtraction to total and residual bin values; it also uses logarithmic addition when considering contributions from several developments [3]. Irish EPA NG4 forms a background statistic by arithmetically averaging measured L_{AF90} values [4]. These operations are not equivalent merely because their operands are expressed in decibels. Conversely, some apparently similar operations are exact or complete in their own terms: compatible equivalent levels add and subtract through their underlying mean energies, and a criterion stated in a percentile can be compared with a measurement of that same percentile without converting it into an equivalent level.

The distinction has long been visible in the acoustics literature, but usually through particular distributional models or particular combination problems. Nelson treated the combination of separate time-varying noise sources by combining discretized source distributions and also tabulated percentile increments for Gaussian sources [5]. Driscoll et al. examined statistical relations between energy-equivalent and percentile sound levels for selected environmental and theoretical distributions, including the normal model [6]. Tang compared logarithmic percentile addition with direct combination of recorded noise histories, first for similar statistical structures [7] and then for longer records and diurnal variation [8]. Ng and Tang extended that comparison to records with different statistical structures [9]. These studies distinguish errors in individual percentile levels from errors in the noise climate, $L_{10} - L_{90}$. More recently, Tonin analysed logarithmic addition and subtraction of statistical noise levels and documented the errors this produced in the cases studied [10].

Related probability and risk-aggregation work treats several richer information classes. Klass and Nowicki [11] use complete marginal laws in quantile bounds for sums, while Blanchet et al. [12] give convolution bounds for complete marginals. Li et al. [13] bound combined quantile integrals under moment information. Bernard et al. [14] treat unimodal distributions, with the nonnegative case corrected in [15]. These results are close to the acoustic problem but do not answer the reporting question considered here. A routine noise report often contains only several percentile values, perhaps an equivalent level, perhaps recorded extrema, and sometimes external information such as a source duty cycle. The question is then not which fitted distribution is most convenient, but which values of the acoustic target remain compatible with exactly that information.

The paper formulates that question as one of identification. Following the partial-identification framework used in econometrics [16], a report is treated as information defining a class of compatible records rather than as an implicit distribution model. A target is point identified only when every compatible record gives the same answer. Otherwise the object supported by the report is an identification set. This distinction changes the interpretation of familiar acoustic calculations. A value returned by an arithmetic operation may be exact; it may be a deliberately defined index whose meaning is complete in its own terms; or it may be one value selected from a larger set of physically compatible values. The last case is called unidentified here. The purpose of the analysis is therefore not merely to produce counterexamples to common approximations. It is to determine the complete replacement set, including sharp endpoints, endpoint attainment and the additional information that narrows the set or restores point identification.

The central results replace unsupported background estimates by complete compatible ranges. Equation (9) uses a known total distribution and an evidenced upper bound on source activity. Theorem 4.1 adds a compatible total equivalent level to a reduced percentile-and-support report. Theorem 4.2 states when a separated inactive block recovers the full-window background mean, and Theorem 4.3 separates selection error from a change in background during source operation. Section 4.5 applies this distinction to total-record L_{95} substitution. The analysis does not infer the background from a total equivalent level alone.

The same reporting problem arises when sources are combined or removed. Section 5 gives sharp percentile-arithmetic bounds, the effect of independence and additional component information, certified finite computation, and the distinction between simultaneous source addition and pooling windows. Its principal results remain in the article; the specialist activity, product-root, finite-reduction and dependence proofs are in the Technical Mathematical Supplement, supplied as separate online Supplementary Material and called the Supplement below.

Section 2 fixes the measurement and quantile conventions. Section 3 provides the energy bounds needed for the background analysis in Section 4. Sections 6 and 7 interpret the results for acousticians, standards committees and measurement reports. Section 8 separates within-record identification from transfer to another period or place; Section 9 gives conclusions and future work. The results concern population laws or complete windows, not finite-sample confidence, and assume nonnegative additive energies rather than coherent pressure addition. Sections S.1 to S.8 contain the specialist proofs and extensions, and Section S.9 provides the consolidated notation register.

2. Setting and conventions

2.1. Levels, energies, and quantiles

Let L be a sound pressure level with specified frequency weighting, time weighting or integration interval, and acoustic reference, fixed within each comparison. We write $X = 10^{L/10}$ and use "energy" for this dimensionless mean-square pressure ratio. Zero energy corresponds to the extended level $-\infty$. Numerical level constants are rounded to four decimals; examples use the precision stated. An atom has positive probability, corresponding to a positive duration fraction in a record; a gap is a value interval of zero probability.

For a scalar variable Z , let $F_Z(z) = \Pr(Z \leq z)$. Its lower and upper generalized quantiles are

$$q_Z(u) = \inf \{ z : F_Z(z) \geq u \}, q_Z^+(u) = \inf \{ z : F_Z(z) > u \}, 0 < u < 1. \quad (1)$$

We use the lower quantile in equation (1) unless specified otherwise. The inverses differ at no more than countably many ranks. We call a reported quantile value an anchor. For finite z , an exact anchor $q_Z(u) = z$ requires $F_Z(z) \geq u$ and $F_Z(x) < u$ for every $x < z$; these conditions are also sufficient. The weaker brackets $Pr(Z < z) \leq u \leq Pr(Z \leq z)$ may also hold within a gap in the support and at its upper endpoint.

ISO 1996-1:2016 defines the N percentage exceedance level for time-weighted and frequency-weighted sound pressure levels [17], Sec. 3.1.3. For the chosen level distribution, we specify its interpretation at jumps and plateaus through the lower quantile. Writing $Q = q_L$ and $q = q_X$, we have

$$q(u) = 10^{Q(u)/10}, L_N = Q(1 - N/100), 0 < N < 100.$$

Thus L_{90} denotes the lower quantile at rank 0.10, and

$$Pr(L > L_N) \leq N/100 \leq Pr(L \geq L_N).$$

Both inequalities can be strict when the level distribution has an atom at L_N . Ranks zero and one denote one-sided limits in integrals, rather than additional percentile observations.

On an averaging basis compatible with the equivalent continuous sound pressure level [17], Sec. 3.1.6,

$$M = E[X] = \int_0^1 q(u) du = \int_0^1 10^{Q(u)/10} du, L_{eq} = 10 \log_{10} M. \quad (2)$$

The inverse relation $q(u) \leq x \Leftrightarrow u \leq F_X(x)$ gives q the distribution of X under uniform rank measure, proving the integral identity in equation (2). Appendix A provides the quantile arguments. The integrals admit extended nonnegative values; a finite equivalent level requires $0 < M < \infty$. The retained and tail integrals listed in Section S.9 of the Supplement are obtained from standard integrated quantile functions [18], Sec. 2.2.

ISO 1996-2:2017 permits percentile calculation from short-interval equivalent levels or sampled time-weighted levels [1], Sec. 9.3.2.4. Compatibility means that the percentile distribution and equivalent level represent the same energy observations and averaging weights. Contiguous short-interval equivalent levels, weighted by their durations, satisfy this requirement. For time-weighted percentile samples, applying equation (2) to a separately integrated equivalent level requires justification of that same-energy relation.

2.2. Additive energy and activity

For background attribution, the total, background, and source energies satisfy

$$X = Y + S, S \geq 0, Y \geq \beta \geq 0, \quad (3)$$

on a common probability space, where β is a declared background floor. Equation (3) assumes that pressure cross terms vanish on the adopted averaging basis; coherent pressure addition lies outside this model.

Section 5 treats totals formed from component sources, $X = \sum_{i=1}^m X_i$.

Let $A = \{S > 0\}$, $\pi = \Pr(A)$, and $0 \leq \pi \leq \alpha \leq 1$. Then $c = 1 - \alpha$ is a guaranteed inactive fraction, whereas $c_0 = 1 - \pi$ is the actual fraction. The bound α is supported by external evidence, such as an operating log. For an intermittent source, it bounds the duty cycle; the inactive observations themselves may remain unknown. A gap in observed levels establishes neither inactivity nor separation. Background energy and source activity may be dependent; Section 4 states the assumptions needed for recovery. Independence, wherever assumed, means mutual independence. The activity bound must apply on the energy observation basis: a source running for one second in every one-minute integration cell has operating duty cycle $1/60$ but a positive contribution in every cell, so its activity probability on that cell basis is one. Operating logs must therefore be aligned with the observations before supplying α .

2.3. Reports and identification

A report R comprises descriptors and measurement conditions. Its percentile package is

$$P = (u_j, q_j) : j = 1, \dots, k, 0 < u_1 < \dots < u_k < 1, q_j = q(u_j). \textcolor{red}{\circ}$$

The anchors q_j are energies obtained from reported level values. Optional information includes justified support bounds $a \leq X \leq b$, mean energy M , and an evidenced activity bound. In bounded background-attribution classes, we write $0 \leq \beta \leq a \leq b < \infty$, tightening a weaker lower support bound to the background floor when necessary.

Following the partial identification literature in econometrics [16], we study all target values compatible with the reported information and declared assumptions. Let M be the declared class of joint laws and $M(R)$ its subclass satisfying every reported equality and declared bound. Support and activity bounds need not be attained. For a target functional θ , define

$$I_\theta(R; M) = \{\theta(P) : P \in M(R)\}. \quad (4)$$

In equation (4), a singleton gives point identification; every compatible law yields that value. Multiple values preclude point identification; an empty set indicates incompatibility.

An operation is *exact* if it returns a point-identified target, a *valid index* if it is used in its defined assessment terms, and *unidentified* if it returns a single value for a target that admits several compatible values. That value need not itself be compatible. The supported result is the identification set; any claim of greater accuracy requires additional assumptions. Sections 3 to 5 give the largest possible overestimation and underestimation, or state that no finite bound holds.

A bound is sharp if it is the infimum or supremum over the compatible class; it is attained if some compatible law achieves it. Interval brackets specify inclusion. Additional constraints can change sharp bounds. Identification intervals describe what the reports determine, whereas confidence intervals address sampling uncertainty.

A marginal law describes one component alone. A coupling is a joint law with specified marginals: for records, one possible way the components vary together in time. The product of the marginals is the coupling under mutual independence.

2.4. Observation and reporting scope

The results concern a population law or a complete observation window. For a finite record of interval energies, normalized durations $T_i / \sum_j T_j$ define that distribution. The record extrema bound this law; population support bounds require separate justification. Inference from a statistical sample to a population requires sampling assumptions and is not treated here. Section 8 distinguishes the measured window from the reference period.

Extremizing joint laws may split the probability mass at one observed energy among different background and source decompositions. Fixed indivisible record cells instead impose the discrete allocation constraints specified in Appendix B. The formulas are stated for exact anchors. Rounded or classed outputs specify energy intervals I_j with $q(u_j) \in I_j$, determined by the reporting convention and transformed class limits. By definition (4), their identification set is the union of the exact-anchor sets over all compatible anchor vectors. Sharp exact-anchor bounds do not by themselves justify evaluating the endpoints of this union by substituting class limits; that reduction requires a separate argument. ISO 1996-2:2017 requires the measurement basis and class interval to be reported [1], Secs. 9.3.2.4 and 13 d). Section 7 specifies the class-width information needed to form these constraints.

Table 1 gives the measurement reading of the quantities used throughout, and Table S9.1 of the Supplement collects the complete notation.

Table 1. The quantities of this paper in measurement terms. A level is a decibel value and the symbol beside it is the energy whose logarithm that level is, by equation (2); the two are never the same quantity and the table keeps them apart.

Symbol	What it means in measurement terms	Defined in
L_{eq}, M	the equivalent continuous level of the record, and the mean sound energy behind it	Section 2.1, equation (2)
$L_N, q(u)$	the percentile levels the report gives, such as L_{90} , L_{95} or L_{10} , and the energies behind them	Section 2.1, equation (1)
Y	the residual/background energy component on the common observation basis, defined at every observation so that $X = Y + S$, including observations during which the source is active	Section 2.2, equation (3)
S	the energy of the sound contributed by the source under investigation	Section 2.2, equation (3)
β	an energy floor claimed beneath the residual sound, with the evidence for it	Section 2.2
α	the upper fraction of energy-observation cells or duration mass with a positive source contribution, evidenced on that same observation basis	Section 2.2
I_θ	the range of values of the target quantity that the measurement cannot tell apart, which replaces a single answer	Section 2.3, equation (4)

Symbol	What it means in measurement terms	Defined in
a, b	the lowest and the highest relative energy the record can contain, with the justification for each; their levels are the support bounds in decibels	Section 2.3
E_-, E_+	the sharp lower and upper bounds on mean energy consistent with the percentiles and support bounds; their levels $10 \log_{10} E_-$ and $10 \log_{10} E_+$ bound the L_{eq}	Section 3.2, equation (5)
M_Y	the full-window mean energy of the residual/background component; its level $10 \log_{10} M_Y$ is the full-window residual equivalent level, distinct in general from the mean over observed source-off periods	Section 4.1
M_S	the mean energy of the source, whose level $10 \log_{10} M_S$ is the source L_{eq} , the level that would be measured from the source alone	Section 4.1
m	the number of sources combined	Section 5
η_i	the energy anchor of the percentile one source or one window reports, at the rank everything is compared at	Sections 5.1 and 5.8
η_Y, x	the energy anchors of the background percentile and the total percentile, at the same rank	Section 5.3
p	the rank in use, $1 - N / 100$ for L_N ; 0.10 for L_{90} and 0.05 for L_{95}	Section 2.1
e_p	the smallest combined percentile energy possible when independent sources each report that rank	Section 5.2, equation (16)
Λ	the energy obtained by adding a source equivalent level to a background percentile	Section 5.3
$D_p^{(m)}$	the error of adding component percentile levels in energy	Section 5.1

3. What a percentile package identifies about the equivalent level

3.1. Identical percentiles, different equivalent levels

Percentile levels are routinely used to say something about the equivalent level: through fixed offsets, through the Gaussian conversion of ISO 1996-2:2017, Annex I.2.2 [1], or by reading a low percentile as a background equivalent level. A finite percentile package cannot support any of these uses without further information.

Consider a night hour of steady 35 dB sound, and the same hour with one 18-second event at approximately 98.01 dB. In energy relative to the steady level, the first record has energy 1 throughout; the second has energy 1 for a fraction 199 / 200 of its duration and energy 1999801 for the remaining 1 / 200. Both report $L_1 = L_{10} = L_{50} = L_{90} = L_{99} = 35$ dB.

For the second record, $F_x(1) = 199 / 200 > 0.99$; both quantile conventions therefore give energy 1 at every reported rank, including the largest. Its mean energy is

$$\frac{199}{200} + \frac{1999801}{200} = 10000.$$

The equivalent levels are consequently 35 dB and 75 dB (Figure 1): one event of 18 seconds raises the equivalent level by 40 dB while every reported percentile stays at 35 dB. More generally, without an upper support or tail constraint, energy above the largest reported rank can increase without altering any reported quantile. The equivalent level is then unbounded above, even for a fixed feasible package.

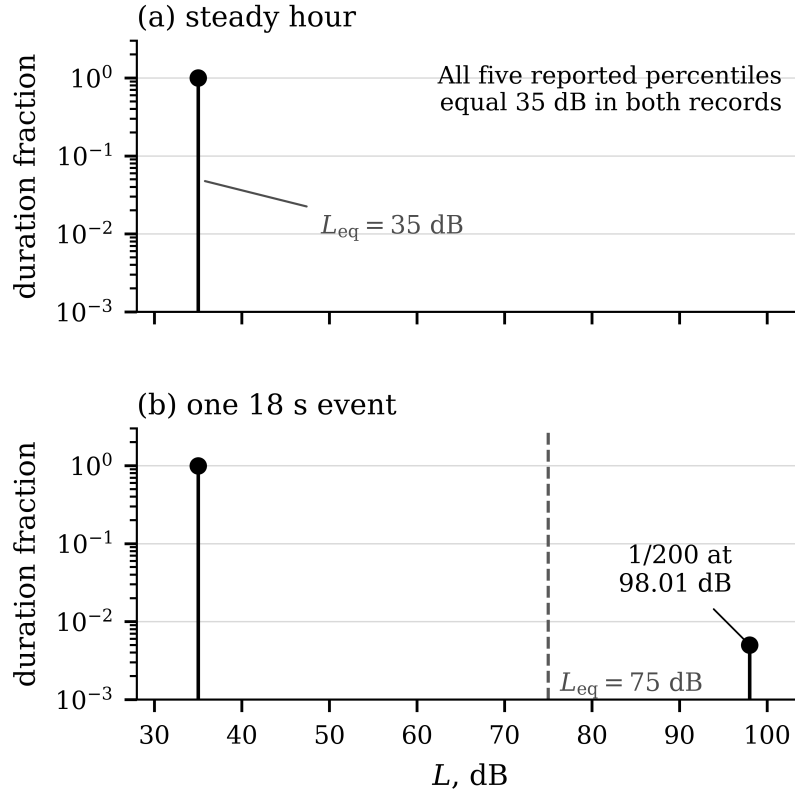


Figure 1. Exact distributions from Section 3.1. Horizontal position gives level; vertical position gives duration fraction on a logarithmic scale. In panel (a), the equivalent level is 35 dB and coincides with the single recorded level, so it is identified by a leader label. In panel (b), a dashed line marks the equivalent level of 75 dB. The second record places duration fractions $199 / 200$ and $1 / 200$ at 35 dB and approximately 98.01 dB. The reported L_1 , L_{10} , L_{50} , L_{90} and L_{99} all equal 35 dB under both quantile conventions.

The discrepancy concerns the complete records themselves. It therefore persists without sampling uncertainty or measurement error: the reported percentiles omit information needed to determine the energy integral.

3.2. What percentiles and the recorded extremes determine

On a common observation basis, the record extrema and reported percentiles give sharp bounds on its equivalent level. A maximum obtained with a different time weighting requires separate justification.

Bertsimas and Popescu [19], Sec. 2, formulate probability bounds under moment constraints. That formulation does not establish the exact-quantile endpoint rules needed here: our constraints are quantiles and support bounds, and our objective is mean energy. We use the quantile representation in Section 2.1.

Suppose the information consists only of a feasible exact package and support bounds $0 \leq a \leq X \leq b < \infty$, the lowest and the highest relative energy the record can contain. The energy anchors satisfy $a \leq q_1 \leq \dots \leq q_k \leq b$. Set $u_0 = 0$, $u_{k+1} = 1$, $q_0 = a$, $q_{k+1} = b$, and $\Delta u_j = u_{j+1} - u_j$. To obtain energy bounds, multiply each rank gap by its lower or upper energy anchor and sum.

Theorem 3.1 (Finite-Quantile Energy Bounds). Under these assumptions,

$$E_- = \sum_{j=0}^k \Delta u_j q_j \leq M \leq E_+ = \sum_{j=0}^k \Delta u_j q_{j+1}. \quad (5)$$

The upper endpoint is attained. The lower endpoint is attained if and only if every reported anchor equals a ; otherwise it is an unattained infimum. Every strictly intermediate value is attained.

Equation (5) gives the closure of the mean-energy set; the theorem supplies its endpoint inclusion. Applying $10 \log_{10}$ preserves that inclusion, with the extended value at zero. Recorded extrema serve here as support bounds. Requiring a finite record to reproduce observed endpoint masses or fixed cell weights can narrow the class and change attainment (Section 2.4).

Let l and h equal q_j and q_{j+1} on each gap (u_j, u_{j+1}) . Their integrals are E_- and E_+ . Monotonicity places every compatible quantile between them almost everywhere. The upper completion is attainable; short rises immediately before the anchors approach the lower bound. Left continuity determines whether that bound is reached. Appendix A gives the construction and proves that interpolation fills the interval.

For example, support bounds $[1, 3]$ and median energy 2 give the exact mean-energy set $(3/2, 5/2]$. Its level image has endpoints approximately 1.76 dB and 3.98 dB, open below and closed above. Equal masses at energies 2 and 3 attain the upper endpoint.

3.3. Which percentiles to report

On a declared bounded support, each additional exact percentile can only narrow the mean-energy interval. Without upper support or other tail control, even arbitrarily fine finite percentile reporting need not bound the mean. Changing anchor values and adding ranks have different effects. For a retained rank $u \in [0, 1]$, the step bounds are

$$\begin{aligned} E_u &= \sum_{j=0}^k \max\{0, \min(u, u_{j+1}) - u_j\} q_j, \\ \bar{E}_u &= \sum_{j=0}^k \max\{0, \min(u, u_{j+1}) - u_j\} q_{j+1}. \end{aligned} \quad (6)$$

They are the sharp infimum and supremum of $\int_0^u q(s)ds$. The upper endpoint is attained; the lower endpoint is attained precisely when every anchor with $u_j \leq u$ equals a . Every intermediate value is attained.

Lemma 3.2 (Monotonicity of the step bounds). Fix the ranks and support bounds. For every $u \in [0, 1]$, both bounds in equation (6) are nondecreasing in each anchor. Thus increasing anchors coordinatewise, while preserving their admissibility, cannot decrease either bound.

Proof. Every anchor coefficient in equation (6) is nonnegative. Subtraction of the corresponding sums proves the assertion. $\square$

Lemma 3.2 concerns exact anchor vectors; for rounded reports, the union interpretation of Section 2.4 applies.

Adding an exact anchor instead raises the lower step function and lowers the upper one. Hence E_- cannot decrease, E_+ cannot increase, and the decibel width $w_{dB} = 10 \log_{10}(E_+/E_-)$ cannot increase wherever it is finite. Figure 2 illustrates the resulting level intervals.

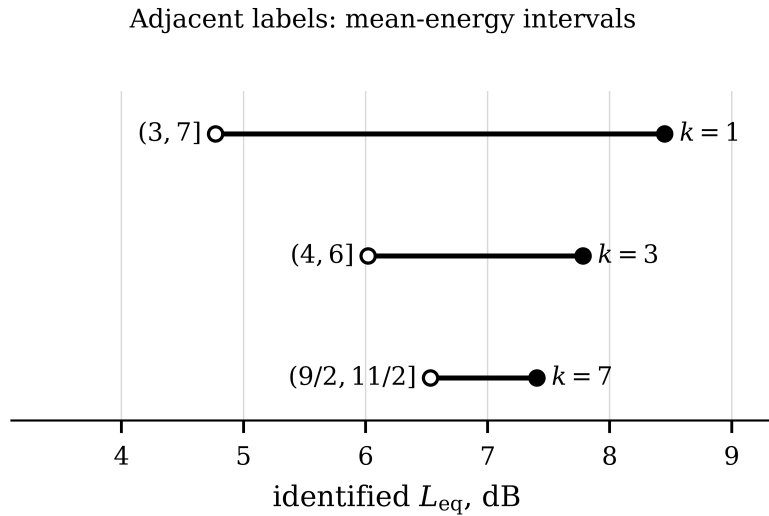


Figure 2. Exact identified level intervals under percentile refinement, calculated from Theorem 3.1. The support bounds are $[1, 9]$; anchors are taken from $q(u) = 1 + 8u$ at $u_j = j/(k+1)$, for $k = 1, 3, 7$. The adjacent labels give mean-energy sets $(3, 7]$, $(4, 6]$, and $(9/2, 11/2]$; the horizontal positions are their decibel transforms. Open and filled circles indicate excluded and included endpoints.

For fixed ranks, the energy width satisfies

$$E_+ - E_- = \sum_{j=0}^k \Delta u_j (q_{j+1} - q_j) \leq \Delta_{\max} (b - a). \quad (7)$$

Here $\Delta_{\max} = \max_j \Delta u_j$. Equality is attained by taking every anchor at a up to a largest rank gap and at b thereafter. Equation (7) therefore gives the exact worst-case energy width. When $a < b$, equal gaps uniquely minimize this worst-case width for k reported ranks, yielding $(b-a)/(k+1)$. This reporting criterion

concerns additive energy width, not decibel width. When $a < b$, every finite package has positive width in this class.

A common finite support bound ensures convergence as the largest gap tends to zero. Without common tail control, increasingly fine finite rank grids alone are insufficient. Let X equal 1 with probability $k/(k+1)$ and $(k+1)^2$ otherwise. Every anchor at $j/(k+1)$, $1 \leq j \leq k$, equals 1, yet

$$M = \frac{k}{k+1} + k + 1 \rightarrow \infty.$$

3.4. Fixed offsets and same-descriptor comparisons

Comparing L_{90} between records on the same measurement basis establishes the ordering of those percentile levels. Comparison with a criterion expressed in that same descriptor is likewise well defined: it is a valid index. Neither comparison determines the difference between mean energies. The difficulty begins when a percentile is used as an energy quantity.

Proposition 3.3 (Single-Percentile Mean Bound). Let $0 < p < 1$. For nonnegative energy constrained only by the positive anchor $q(p)$, the exact set of finite means is $((1-p)q(p), \infty)$. Every value is attained by a bounded law. Consequently, with $p = 1 - N/100$,

$$L_N < L_{eq} + 10 \log_{10} \frac{100}{N}.$$

Appendix A.2 proves Proposition 3.3. The constant is a sharp, unattained supremum: 0.4576 dB for L_{90} . The equivalent level can exceed that percentile by any amount. For L_{10} the constant is 10 dB. The offset $L_{10} \approx L_{eq} + 3$ dB of Driscoll et al. [6], Secs. I to III and App. A, obtained from selected environmental and theoretical distributions and an analysis of normal distributions, therefore describes those distributions rather than sound levels in general: an equivalent level estimated as $L_{10} - 3$ dB can be high by less than 7 dB and low by any amount. Converting a percentile to an equivalent level by a fixed offset is unidentified from the report. With a justified support bound, Theorem 3.1 gives the replacement interval. A tail restriction is another route to a finite upper bound. She, Tang and Wang [20] impose bounded residual surplus on a nonnegative variable V , equivalently the new-better-than-used-in-expectation condition that the conditional mean excess above every level with positive survival probability does not exceed the overall mean. When a fraction $1-p$ of the distribution lies at or above a value v_0 , their Lemma 7 gives the sharp compatibility range $v_0/(1 - \ln(1-p)) \leq E[V] < v_0/p$. Their observation is a survival probability at a specified value, not the exact lower-quantile anchor used here. If the same tail condition is imposed on the nonnegative energy variable X with $M < \infty$, the present anchor $q(p)$ gives directly $M \leq q(p)/p$: for every $s > q(p)$, $Pr(X \geq s) \leq 1-p$ and bounded residual surplus gives $E[(X-s)^+] \leq M Pr(X \geq s) \leq (1-p)M$. Since $X \leq s + (X-s)^+$, it follows that $M \leq s + (1-p)M$, and letting $s \downarrow q(p)$ gives the bound. The absence of a finite upper bound in Proposition 3.3 is therefore a consequence of declaring no tail restriction, not of the single anchor.

The same bound governs a contrast between a source equivalent level $L_{eq,s}$ and a background percentile $L_{N,y}$. Proposition 3.3 applied to the background gives

$$L_{eq,S} - L_{eq,Y} < (L_{eq,S} - L_{N,Y}) + 10 \log_{10} \frac{100}{N},$$

with no bound in the other direction. Such a contrast is a valid index in its own terms, and unidentified as an estimate of the contrast between source and background equivalent levels: at a fixed contrast the background equivalent level is unbounded above. Read as an index, the criterion must be stated as a source equivalent level against a background percentile, not as a difference of equivalent levels.

Attribution requires additional information. Under the additive model of Section 2.2, assume finite M , zero background floor, and an externally justified activity bound α . Write $J_c = \int_0^c q(u) du$ for the whole-window energy contribution of the quietest fraction c . With $c = 1 - \alpha$ and $0 < p < c$, the lower-tail selection bound developed in Section 4 gives

$$E[Y] \geq J_c \geq (c - p)q(p), E[S] \leq M - (c - p)q(p).$$

Thus a total percentile, together with the activity bound, supplies a background-energy lower bound and a source-energy upper bound. It does not recover either component mean. Section 4 develops the sharp attribution intervals and the effect of a positive background floor.

3.5. The Gaussian conversion of ISO 1996-2 Annex I.2.2

ISO 1996-2:2017, Annex I.2.2 [1], estimates the equivalent level of the residual sound from L_{50} and L_{90} or L_{95} through Gaussian formulas. Suppose the sound-level distribution is Gaussian. Evaluating its exponential moment gives

$$L_{eq} = E[L] + \frac{\ln 10}{20} E[(L - E[L])^2].$$

Let Z be standard normal. Gaussian symmetry yields

$$L_{eq} = L_{50} + \frac{\ln 10}{20} \left(\frac{L_{50} - L_N}{q_Z(N/100)} \right)^2, N \in \{90, 95\}. \quad (8)$$

Its Formulae (I.1) and (I.2) use numerical approximations 0.115, 1.28, and 1.65 in place of $\ln 10 / 20$, $q_Z(0.90)$, and $q_Z(0.95)$. The annex attributes the formulas to NF S 31-010. The same Gaussian identity is set out in [6], App. A, which gives the equivalent level of a normal level distribution as $L_{50} + 0.115\sigma^2$ and its percentile levels as $L_5 = L_{50} + 1.65\sigma$ and $L_{10} = L_{50} + 1.28\sigma$, the same constants that Annex I.2.2 applies to L_{95} and L_{90} through the symmetry of the normal distribution. The annex's Gaussian assumption concerns residual sound, with total sound described as bimodal. Applying equation (8) to total-sound percentiles therefore requires a separate justification.

Equivalently, for two percentiles of the same Gaussian distribution,

$$L_{eq} = \frac{L_{10} + L_{90}}{2} + \frac{\ln 10}{80 [q_Z(0.90)]^2} (L_{10} - L_{90})^2.$$

The energy distribution is then lognormal. A nondegenerate Gaussian level has unbounded support, so this assumption defines a different class from Theorem 3.1; a constant level satisfies the identities with zero variance. Appendix A provides the derivation.

For a fixed percentile-based prediction, define error as the predicted level minus the true level, consistently with Table 4. The possible errors are obtained by subtracting each level in the identified set from the prediction, preserving endpoint inclusion. Without tail control, the error can be unbounded below, as Section 3.1 demonstrates.

Applied to the two records of Section 3.1 read as records of residual sound, which report $L_{50} = L_{90} = 35$ dB, equation (8) returns 35 dB for both, while the second record's equivalent level is 75 dB.

Equation (8) is exact under the Gaussian assumption for the sound-level distribution. The two reported percentiles alone do not establish that assumption. Without that model, justified support bounds give the interval in Theorem 3.1. Without an upper support bound or another tail restriction, the compatible equivalent levels are unbounded above.

3.6. Implications for measurement practice

When the full compatible record is retained, compute its equivalent level by energy averaging. When only percentiles survive, report the interval supported by those anchors and justified same-basis extrema; additional ranks can narrow it. A fixed offset or an unverified Gaussian conversion supplies no replacement for that missing information. A criterion stated in the measured percentile remains directly assessable.

4. Background and source from the total record

4.1. Identification from the total distribution

Environmental-noise assessments do not always include a measurement with the source switched off. Practice may then use a low percentile of the total record, such as L_{90} , as a background descriptor; ISO 1996-2:2017 Annex I.2.1 specifically proposes L_{95} under its stated condition [1]. The background equivalent level may also be taken as the mean of the quietest part of the record. Sections 4.1 to 4.4 establish what the total record identifies without either substitution; Section 4.5 then tests the percentile substitution against those results.

We retain the additive energy model of Section 2.2, with finite total mean M , declared floor β , and externally supported activity bound $\Pr(S > 0) \leq \alpha$. The total distribution and equivalent level use the same observations and weights. We first suppose that the total law is known. The target residual mean is $M_Y = E[Y]$ over the full observation basis, and the target source mean is $M_S = E[S]$, with identification sets I_Y, I_S . Their levels are $10 \log_{10} M_Y$ and $10 \log_{10} M_S$. A mean measured only during source-off periods is a conditional quantity and need not equal M_Y without an additional recovery condition. Neither independence nor separation is assumed.

The largest admissible source mean is the energy in the highest α fraction of the total distribution, less the background floor on that fraction. With $c = 1 - \alpha$, the complete mean-energy intervals are

$$I_Y = [J_c + \alpha\beta, M], I_S = [0, K_\alpha], K_\alpha = M - J_c - \alpha\beta. \quad (9)$$

Every pair satisfying equation (9) and $M_Y + M_S = M$ is attained, including the endpoints. Appendix B.1 gives the construction, allowing probability mass at a single total energy to be divided between component allocations. If instead $Y > 0$ is required, with $X > 0$ almost surely, $\beta = 0$ and $\alpha > 0$, the background lower endpoint and source upper endpoint are excluded.

The upper-tail construction proves the interval is sharp; it does not label the source-active observations. Different temporal allocations can give the same component means.

Tail-mean optimization has an established threshold formulation [21], Sec. 2, Theorem 1, stated there for continuous loss distributions. Appendix B.1 supplies the selection argument with atoms. Ordered couplings, which pair background and total energies with background no greater than total, can preserve the common part of two marginal laws on the diagonal [22], Proposition 2.6. This construction maximizes the probability of equality [23], Sec. 2. Consequently, a candidate background law supported above β is compatible precisely when it is stochastically below the total law and their total-variation distance is at most α . That distance is the minimum activity over ordered couplings; the criterion establishes existence, rather than validating a fixed temporal pairing.

For fixed indivisible record cells, Appendix B.6, Proposition B.4, gives the corresponding selection result; the probability-law bounds remain valid outer bounds.

4.2. Percentile reports with a known total mean

Suppose the available information is an exact percentile package, support bounds $0 \leq \beta \leq a \leq X \leq b < \infty$, the activity bound, and a feasible total mean M . Section 3.2 defines the step functions l, h ; equation (6) gives their partial integrals. Set $U_{tail} = \bar{E}_1 - \bar{E}_c$. Without a reported mean, these data give the sharp background interval with closure $[\alpha\beta + \underline{E}_c, E_+]$ and source interval $[0, U_{tail} - \alpha\beta]$. The background lower endpoint is attained exactly when every reported anchor at a rank no greater than c equals a ; the other endpoints are attained. Appendix B.2 includes this specialization.

When the total mean is reported, it must constrain the same quantile completion as the anchors. The smallest retained integral must satisfy both the percentile lower bound and the total mean minus the largest permitted tail integral.

Feasibility of the reported mean is checked against Theorem 3.1: $E_- < M \leq E_+$ unless all reported anchors equal a , in which case $E_- \leq M \leq E_+$. An incompatible report defines an empty class; refinement cannot repair it.

Li et al. [13] bound quantile integrals under moment constraints; Bernard et al. [14] treat unimodal distributions, with the nonnegative case corrected in [15]. Bignozzi and De Vecchi [24], Sec. 2.2, specify an entire lower quantile segment and the mean. Theorem 4.1 instead uses finitely many exact anchors, support bounds and the total mean, without shape restrictions.

Theorem 4.1 (Total-Energy Refinement). For this information class,

$$\begin{aligned} \underline{E}_c^{(M)} &= \max\{\underline{E}_c, M - U_{tail}\}, \\ \bar{T}_Y &= [\alpha\beta + \underline{E}_c^{(M)}, M], \\ \bar{T}_S &= [0, M - \alpha\beta - \underline{E}_c^{(M)}]. \end{aligned} \tag{10}$$

The background upper endpoint and zero-source endpoint are attained. The opposite endpoints are attained if and only if $M > \underline{E}_c + U_{tail}$ or every reported anchor with $u_j \leq c$ equals a . Every strictly intermediate pair with sum M is attained. Appendix B.2 proves sharpness with the total mean held fixed.

The mean refinement matters when the smallest permitted retained energy and largest permitted tail cannot supply the reported total: the excess must remain in the lower block. The floor contribution remains explicit. If $a > 0$, $\beta = 0$, $\alpha > 0$ and strictly positive background is required, the background lower endpoint and source upper endpoint are excluded irrespective of the anchor condition. These are identification ranges, not confidence intervals.

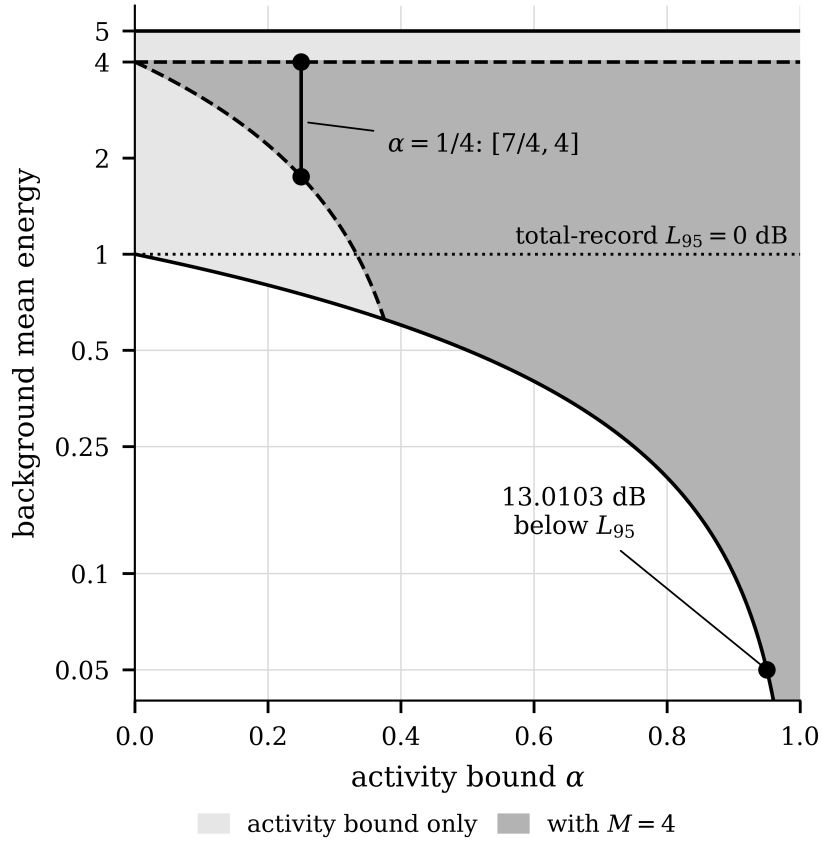
For example, take $1 \leq X \leq 9$, $q(1/2) = 1$, $\alpha = 1/4$ and $\beta = 0$. The percentile constraints give $\underline{E}_{3/4} = 3/4$ and $U_{tail} = 9/4$. Reporting $M = 4$ raises the background lower bound to $\max\{3/4, 4 - 9/4\} = 7/4$, so

$$I_Y = [7/4, 4], I_S = [0, 9/4].$$

This example permits zero background energy on the active observations. The total law assigning probabilities $1/2, 1/4, 1/4$ to energies $1, 5, 9$ has the required median and mean. Assigning the top quarter entirely to the source attains the endpoints. The background lower-bound level increases from $10 \log_{10}(3/4) \approx -1.25$ dB to $10 \log_{10}(7/4) \approx 2.43$ dB. Combining the old lower bound with the newly reported mean would give source energy $13/4$, exceeding the sharp ceiling $9/4$.

Subtract source energy before applying $10 \log_{10}$. The transform preserves positive endpoint inclusion; zero corresponds to $-\infty$, not a finite lower source level.

Figure 3 illustrates the dependence on the activity bound. For rounded anchors, the union of Section 2.4 applies with the mean constraint retained in every package; equation (10) does not justify substituting class limits independently.



At $\alpha = 0.05$: activity bound only, gap 0.2228 dB.

Equation (13): general supremum 0.4576 dB.

Figure 3. Sharp background-energy intervals from the exact report $1 \leq X \leq 9$, $q(1/2) = 1$, $\beta = 0$, plotted against the activity bound on a logarithmic energy ordinate, with and without $M = 4$. All interval endpoints are attained; filled circles mark the worked interval $[7/4, 4]$ at $\alpha = 1/4$ and the lower endpoint at $\alpha = 0.95$. The total-record $L_{95} = 0$ dB is shown for comparison. At $\alpha = 0.05$, without the mean, the package-specific lower endpoint is 0.95, giving a sharp gap of approximately 0.2228 dB. With $M = 4$, the lower endpoint is $71/20 = 3.55$. The general, unattained 0.4576 dB supremum of equation (13) is valid but not sharp for this package. At $\alpha = 0.95$, corresponding to the exact-inactivity strengthening of the annex's 5% quiet-time condition, the lower endpoint is 0.05, or 13.0103 dB below L_{95} , with or without $M = 4$. The logarithmic display ends before the zero endpoint at $\alpha = 1$.

4.3. The quietest part of the record as background

Taking the mean of the quietest part of the record as the background is exact only under conditions that the level record alone cannot confirm; this subsection states them. An activity bound does not identify the inactive observations or their actual fraction. For recovery, let $0 < \pi < 1$, $c_0 = 1 - \pi$, and suppose that c_0 is known or identified within the declared model. Write $M_{\text{off}} = E[Y \mid A^c]$ and $\delta = E[Y \mid A] - M_{\text{off}}$; the difference δ may have either sign.

Theorem 4.2 (Separated-Block Recovery). Under the additive model with finite M and $0 < \pi < 1$, suppose c_0 is known or identified and the inactive observations form a separated lower block:

$$\text{ess sup}(X \mid A^c) < \text{ess inf}(X \mid A).$$

The essential bounds ignore probability-zero exceptions; the inequality requires a strict gap between the inactive and active energy ranges. Then $M_{\text{off}} = J_{c_0} / c_0$. The whole-window background mean equals M_{off} if and only if $\delta = 0$. In that case,

$$M_Y = \frac{J_{c_0}}{c_0}, M_S = M - \frac{J_{c_0}}{c_0}. \quad (11)$$

If Y is independent of the activity indicator 1_A , the entire background law is recovered through $q_Y(u) = q(c_0 u)$ for $0 < u < 1$. Appendix B.3 proves these statements.

In equation (11), J_{c_0} is energy averaged over the whole window but retained only on its lowest fraction, whereas J_{c_0} / c_0 is the mean within that fraction. Equation (11) requires equality of conditional means; independence of background and activity is sufficient but stronger. Within the separated, independent class, connected background support identifies c_0 as the height of the first nontrivial plateau of F_X : an interval on which the distribution function is constant at a probability strictly between zero and one. A gap between finite sample values does not establish these population assumptions. Replacing c_0 by $1 - \alpha$ generally removes some inactive observations when $\pi < \alpha$. A finite percentile package can leave the retained integral uncertain; exact recovery requires it to be determined.

If operating records identify inactivity, its mean M_{off} is observable without separation. The whole-window target remains $M_Y = M_{\text{off}} + \pi\delta$. The separation assumption is therefore a way to identify the inactive mean from the sorted total distribution, while conditional-mean equality justifies its use for the whole window.

4.4. Recovery error and the identification regimes

Separation and conditional-mean equality have distinct roles. Let B be a set of probability c_0 carrying the lowest total energies, with ties divided when necessary, and define

$$D_{\text{sel}} = E[X 1_{A^c}] - J_{c_0} \geq 0.$$

Theorem 4.3 (Recovery-and-Selection Error). Under the additive model with finite M and $0 < \pi < 1$, without separation or independence,

$$\frac{J_{c_0}}{c_0} - M_Y = -\pi\delta - \frac{D_{\text{sel}}}{c_0}. \quad (12)$$

For $a \leq X \leq b$, the error satisfies

$$-\pi|\delta| - \frac{b-a}{c_0} \Pr(A^c \setminus B) \leq \frac{J_{c_0}}{c_0} - M_Y \leq \pi|\delta|.$$

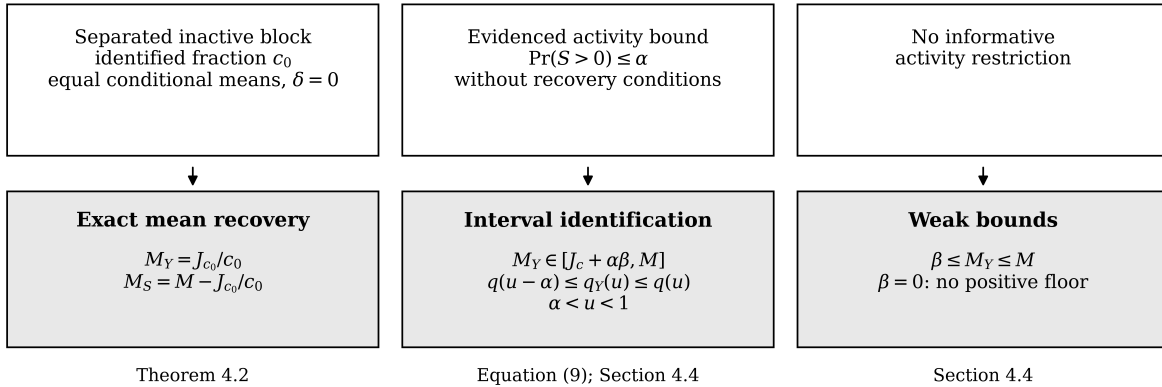
The quietest fraction can misrepresent the background in two ways. Selecting observations other than the source-off observations contributes the downward term $-D_{\text{sel}}/c_0$. A change in background during source operation contributes $-\pi\delta$, of either sign. Appendix B.4 proves equation (12) and the bounds. Exact recovery follows when both terms vanish; separation guarantees the vanishing selection defect. Their cancellation can also produce equality, so separation is not a necessary condition for every possible recovery model. Operational error bounds require justified bounds on the conditional-mean difference and misclassified mass; the total distribution alone generally determines neither.

Three information regimes follow (Figure 4). A separated inactive block with identified fraction and equal conditional background means gives exact mean recovery. An activity upper bound without those recovery assumptions gives the intervals in equations (9) or (10), even when inactive observations are unlabeled. It also bounds background percentiles by

$$q(u - \alpha) \leq q_Y(u) \leq q(u), \alpha < u < 1.$$

Appendix B.5 proves this bracket. With no informative activity bound, the unrestricted nonnegative class gives only $\beta \leq M_Y \leq M$; when $\beta = 0$, no positive lower bound follows. These regimes concern declared information and assumptions, rather than classifications established by a histogram alone.

Known total law, finite M ; $X = Y + S$, $Y \geq \beta \geq 0$, $S \geq 0$



Theorem 4.3 also allows equality through cancellation of $-\pi\delta$ and $-D_{\text{sel}}/c_0$.

Figure 4. Three background-identification regimes for a known total law with finite mean under the additive model and declared background floor. A separated inactive block with identified fraction and equal conditional background means gives the exact-recovery regime of Theorem 4.2. An activity upper bound without those recovery conditions gives equation (9) and the percentile bracket for $\alpha < u < 1$. With no informative activity restriction, only $\beta \leq M_Y \leq M$ remains. Theorem 4.3 also permits equality through cancellation of selection and background-shift errors. Where only a percentile package and total mean are known, equation (10) supplies the separate refinement.

4.5. The L_{95} substitution of ISO 1996-2 Annex I.2.1

To extract the source equivalent level, residual correction subtracts background mean energy from total mean energy. Using the total-record L_{95} as the background input instead gives $M - q(0.05)$. The question is whether that percentile energy represents M_Y .

ISO 1996-2:2017, Annex I.2.1 [1], proposes L_{95} when the source "does not contribute significantly" during "5 % or more of the total measurement time", assuming that this level represents residual sound. The clause supplies neither exact inactivity nor a numerical tolerance for insignificant contribution.

First model the stated time fraction by requiring exact inactivity during at least 5% of the window, which strengthens the qualitative condition. With $\beta=0$, equation (9) at $\alpha=0.95$ gives $J_{0.05} \leq M_Y \leq M$, with both endpoints attained. For $q(0.05) > 0$, left continuity and monotonicity give $0 < J_{0.05} \leq 0.05 q(0.05)$. Under the annex's stated condition, the total-record L_{95} is therefore compatible with background equivalent levels from at least $10 \log_{10} 20 \approx 13.0103$ dB below it up to the total equivalent level. A declared floor raises the attained minimum to $J_{0.05} + 0.95 \beta$ (Appendix B.5).

If Theorem 4.2 additionally applies with $\delta=0$, known $c_0 > 0.05$, and positive $q(0.05)$, then

$$M_Y = \frac{J_{c_0}}{c_0} \geq \frac{c_0 - 0.05}{c_0} q(0.05), 10 \log_{10} \frac{q(0.05)}{M_Y} \leq 10 \log_{10} \frac{c_0}{c_0 - 0.05}.$$

There is no finite bound in the opposite direction without further support restrictions. At $c_0 = 0.05$, under the same recovery assumptions, $M_Y \leq q(0.05)$, with equality exactly when the inactive energy is almost surely constant. With $\beta=0$ and no further support restrictions, an unknown $c_0 \in [0.05, 1)$ permits either error sign and unbounded error in both directions, even under these recovery assumptions. Appendix B.5 supplies the derivation and witnesses. Under the annex's own condition, the total-record L_{95} therefore carries no error bound in either direction over this class of records unless further support restrictions are declared.

A different, substantially stronger assumption is exact inactivity during at least 95% of the window, corresponding to $\alpha=0.05$. It requires external justification and does not follow from the annex's 5% condition.

For $0 < \alpha < 1/2$ and $q(\alpha) > 0$, left continuity makes $\int_0^\alpha q(u) du$ positive, while monotonicity on $[\alpha, 1-\alpha]$ gives $\int_\alpha^{1-\alpha} q(u) du \geq (1-2\alpha) q(\alpha)$. Combining these facts with equation (9),

$$M_Y \geq J_{1-\alpha} + \alpha\beta > (1-2\alpha) q(\alpha), 10 \log_{10} \frac{q(\alpha)}{M_Y} < 10 \log_{10} \frac{1}{1-2\alpha}. \quad (13)$$

Under nonnegative additive energies, a compatible observation basis and an externally justified exact activity bound $\alpha=0.05$, equation (13) bounds $L_{95} - 10 \log_{10} M_Y$ by $10 \log_{10} (10/9) \approx 0.4576$ dB. It holds for every compatible actual background, without a representativeness or independence assumption, and also bounds the excess above the sharp lower-endpoint level. There is no corresponding finite bound on underestimation over this class. Appendix B.5 establishes that the constant is an unattained sharp supremum and that the opposite direction is unbounded already in the class with $\beta=0$. The annex's condition is a quiet 5% of the window; the 0.4576 dB bound of equation (13) is proved under a quiet 95%. For an exact activity bound $0 < \alpha < 1/2$, equation (13) limits the overstatement by the matching percentile $q(\alpha)$, the level

$L_{100(1-\alpha)}$, to $10 \log_{10} [1 / (1 - 2\alpha)]$ dB: 0.9691 dB at $\alpha = 0.10$, read on L_{90} , and 3.0103 dB at $\alpha = 0.25$, read on L_{75} . At $\alpha \geq 1/2$ equation (13) gives no bound.

Proposition 3.3 gives the same constant at $p = 0.10$ for a percentile and the mean of one distribution, without an activity assumption. Equation (13) instead bounds the total-record L_{95} against the background mean; the equal constants do not identify the two statements.

If insignificant contribution is quantified by $Pr(S > \rho Y) \leq \alpha$, with $\rho \geq 0$ and $\beta = 0$, the lower bound is $M_Y \geq J_c / (1 + \rho)$. Insignificance during at least 5% corresponds to $\alpha = 0.95$ in this model. A numerical bound requires a declared ρ . Section 6 develops the standards comparison.

Substituting percentile energy for background mean energy reverses the error in the source attribution. Under the exact sparse-activity model $\alpha = 0.05$, equation (13) gives the quantitative envelope

$$-\frac{M_Y}{9} \leq (M - q(0.05)) - M_S = M_Y - q(0.05) \leq M_Y.$$

The left inequality is strict when $q(0.05) > 0$; the non-strict enclosure also covers zero energy. Table 2 gives the sign consequences. All four cases occur with exact inactivity during at least 5% of the window; under $\alpha = 0.05$, background overestimation is limited by equation (13), while underestimation remains unbounded in decibels. Percentile substitution therefore does not identify the source mean energy or guarantee an upper bound on it.

Table 2. Attribution consequences of replacing background mean energy by total-record percentile energy $q(0.05)$.

Relation to the actual background	Consequence of using $M - q(0.05)$ for source mean energy
$q(0.05) < M_Y$	Source energy is overestimated.
$q(0.05) = M_Y$	Source mean energy is recovered exactly.
$M_Y < q(0.05) \leq M$	Source energy is underestimated.
$q(0.05) > M$	The subtraction is negative and cannot represent nonnegative source energy.

The substitution of a total-record percentile for the background equivalent level is therefore unidentified. The replacement depends on the information actually retained from the measurement. With the total distribution and an evidenced activity bound, equation (9) gives the background interval. With a finite percentile package, compatible support bounds, the total mean and the activity bound, equation (10) gives the corresponding interval. When the inactive block, its fraction and the recovery conditions of Theorem 4.2 are justified, equation (11) gives exact recovery.

4.6. Implications for measurement practice

A quiet total-record percentile is not generally the background equivalent level. The usable replacement is an interval based on the full record or reduced report, the background floor and externally evidenced

activity. Keep the activity observation basis explicit: a brief event in every integration cell can make every cell source-active.

For exact recovery from a quiet block, justify which observations are inactive, their fraction and why their background mean applies to source-active periods. Theorems 4.2 to 4.3 distinguish that evidence from a visible histogram gap or an accidental cancellation of errors. If instead a background percentile is supplied as a separate input to total-mean subtraction, Proposition 5.4 gives the corresponding source-energy range. That is a different information class from using a percentile of the total record itself.

5. Combining sources

Section 4 addressed source extraction from a measured total, including the substitution of L_{90} or L_{95} for background equivalent level. Here we determine what component reports establish about source correction and combined sound. Logarithmic addition and subtraction of percentiles are used to combine sources and to remove background, for example in wind-farm assessment [10] and in the historical tables of Nelson [5].

Let $X = \sum_{i=1}^m X_i$, with $m \geq 2$, on the common observation basis of Section 2.

Logarithmic addition converts reported levels to energies, sums them, and converts back to decibels. For equivalent levels it is exact, because mean energies add under every coupling:

$$L_{eq} = 10 \log_{10} \sum_i \mu_i, \mu_i = E[X_i].$$

Source subtraction $M_S = M - M_Y$ is exact for compatible means of the same target decomposition. A residual measured in another window needs the transfer justification of Section 8; a quiet subset of this window needs the recovery conditions of Section 4. A percentile supplies neither mean by itself, and the quietest periods of different sources need not coincide. Sections 5.1 to 5.3 give the resulting addition, subtraction and mixed-descriptor sets; Sections 5.4 to 5.7 show how operating records, support information and dependence assumptions can narrow them.

Nelson [5] combines discretized source distributions by multiplying class probabilities (Method 1; [5], Eqs. (5) and (6)), thereby imposing independence. Kessler and Gottlieb [25], Sec. I and Eq. (7), explicitly assume independence and combine sound-level distributions by assigning each pair of input levels its product probability at the level obtained by adding their energies. These methods use distributional information that a percentile package does not retain. Nelson's Gaussian-source tables give L_{10} and L_{90} increments ([5], Tables 1 and 2), and the simple approximations he fitted to the L_{10} increments had no counterpart for L_{90} ([5], p. 10). Outside a fitted distribution class no such relationship exists at either rank: Theorem 5.2 shows that, even under independence, component percentiles at a common rank determine only a lower bound on the combined percentile at that rank.

Tonin [10], Secs. 1 to 3, studies logarithmic addition and subtraction, motivated by wind-turbine assessment, using Gaussian level distributions and combined recordings. Li et al. [13], Sec. 3.2, bound combined quantile integrals under moment information; Blanchet et al. [12], Theorems 2 and A.2, give convolution bounds for complete marginals.

Here component laws vary subject to finite reports. Complete-marginal aggregation bounds can be composed with an outer worst case over the laws compatible with the reports. That composition is valid in general, and sharp wherever the equality cases of Theorems 2 and A.2 apply, two components among them. What distinguishes the treatment here is a closed form in the reported anchors with the attaining laws exhibited, together with the restricted coupling classes, independence in Theorem 5.2 and a dependence budget in Theorem 5.9, that a worst case over all couplings with given marginals does not address. To the best of our knowledge, Theorem 5.2, Theorems 5.5 to 5.7 and Theorem 5.9 are new.

5.1. Adding percentiles: what holds for every overlap in time

Adding the L_{90} values of sources logarithmically is the usual way to predict their combined L_{90} . We first allow any temporal overlap between sources. At a common rank $0 < p < 1$, write $\eta_i = q_{X_i}(p) > 0$ for the percentile each source reports, in energy. The error of logarithmic (energy) addition of component L_N values is

$$D_p^{(m)} = 10 \log_{10} \frac{\sum_i \eta_i}{q_X(p)}.$$

Positive error denotes overestimation of the combined percentile level.

Proposition 5.1 (Any-Dependence Percentile-Sum Bound). For nonnegative components, every coupling satisfies

$$q_X(p) \geq \max_i \eta_i, D_p^{(m)} \leq 10 \log_{10} \frac{\sum_i \eta_i}{\max_i \eta_i} \leq 10 \log_{10} m. \quad (14)$$

Equation (14) follows from $X \geq X_i$ and monotonicity of quantiles. With only the anchors and nonnegativity imposed, the exact set of combined percentile energies is $[\max_i \eta_i, \infty)$ (Appendix C.1). Its lower endpoint is attained; the unbounded upper endpoint means there is no finite lower error bound. The logarithmic sum of two L_{90} values can therefore be up to 3.0103 dB above the combined L_{90} and any amount below it; for two sources, independence does not change either statement (Section 5.2).

Even complete marginal laws and the total mean do not determine the sign. Give both components probabilities $1/20, 1/20, 9/10$ at $1, 2, 8$. One coupling places probabilities $1/20, 1/20, 9/10$ at $(1, 2), (2, 1), (8, 8)$; another places $1/20, 1/20, 1/20, 17/20$ at $(1, 1), (2, 8), (8, 2), (8, 8)$. At $p = 0.10$, their combined quantiles are respectively 3 and 10, whereas the summed component quantiles equal 4. The errors are +1.2494 dB and -3.9794 dB. Both total means equal $147/10$. Appendix C.1 verifies these witnesses.

The recording studies also exhibit different error signs. For similar bell-shaped records, Tang [7], Secs. 4.6 and 5, found slight L_{10} overestimation and larger L_{90} underestimation; the longer-record study [8], Secs. 4.1 and 5, retained that pattern. With different statistical structures, Ng and Tang [9], Secs. 4.1 and 5, instead reported a tendency to underestimate L_{10} . Their noise climate is $L_{10} - L_{90}$, so its error must be distinguished

from the error in either percentile. These findings concern the recordings and level adjustments studied; they do not establish distribution-free error limits.

Tonin [10], Sec. 5, reports errors from -3 to 0 dB for L_{90} and from 0 to 1 dB for L_{10} in the cases studied, with the same sign convention. Equation (14) gives a universal two-source overestimation cap of 3.0103 dB and no finite underestimation cap; it does not explain those observed signs.

If every component also reports a finite mean μ_i , feasibility requires $\mu_i > (1-p)\eta_i$. With no further restrictions, the exact set over all couplings is

$$\max_i \eta_i \leq q_X(p) < \frac{\sum_i \mu_i}{1-p}. \quad (15)$$

The lower endpoint is attained; the upper is approached but never attained (Appendix C.1). Component percentiles determine the sharp lower endpoint, and equivalent levels determine the upper endpoint. In level terms, equation (15) places the combined L_N below the combined equivalent level plus $10 \log_{10}(100/N)$ dB. Together with equation (14), it bounds both error directions, since $D_p^{(m)} > 10 \log_{10}[(1-p) \sum_i \eta_i / \sum_i \mu_i]$. Under independence the same upper bound holds, but need not be sharp. Synchronized component records would determine the coupling itself.

5.2. Adding percentiles of independent sources

How much does independence narrow this range? Klass and Nowicki [11] use complete marginal laws. Pinelis [26], Theorem 2.4 and Corollary 2.7, compares expectations over specified classes of differentiable, nonincreasing convex test functions under first- and second-moment constraints and derives left-tail bounds. The test classes exclude threshold indicators; these results do not in general give an exact extremal quantile. Under independence, the following optimum uses only exact percentiles and nonnegativity. Its lower bound uses the elementary domination $X_i \geq \eta_i 1_{\{X_i \geq \eta_i\}}$; Appendix C.1 establishes the exact quantile boundary and attainment. Component reports alone cannot establish independence.

Theorem 5.2 (Weighted Bernoulli Envelope). Fix $0 < p < 1$ and $\eta_i > 0$. Let B_i be independent Bernoulli variables with $\Pr(B_i = 0) = p$, and set $W = \sum_i \eta_i B_i$ and $e_p = q_W^+(p)$. Among mutually independent nonnegative components satisfying $q_{X_i}(p) = \eta_i$,

$$\min q_X(p) = e_p, \max D_p^{(m)} = 10 \log_{10} \frac{\sum_i \eta_i}{e_p}. \quad (16)$$

The exact set is $[e_p, \infty)$. Bounded two-point component laws attain equation (16). Appendix C.1 proves the bound, its attainment and interval filling. With strictly positive component energies, e_p remains the sharp infimum.

Independence raises the sharp lower bound on the combined level by $10 \log_{10}[e_p / \max_i \eta_i]$ dB. For two sources this gain is zero, since $e_p = \max(\eta_1, \eta_2)$ (Appendix C.1).

For equal anchors $\eta_i = \eta_1$, let $N_{bin} \sim \text{Bin}(m, 1-p)$ and $\kappa_{m,p} = q_{N_{bin}}^+(p)$. Equation (16) gives $\min q_X(p) = \eta_1 \kappa_{m,p}$, so the gain from independence is $10 \log_{10} \kappa_{m,p}$ dB. At $p=0.10$, corresponding to L_{90} , the sharp error constants $10 \log_{10}(m / \kappa_{m,p})$ for $m=2, 3, 4, 5$ are respectively 3.0103 dB, 1.7609 dB, 1.2494 dB and 0.9691 dB. They require equal anchors: for $(\eta_1, \eta_2, \eta_3) = (1, 100, 100)$, $e_{0.10} = 101$, giving 2.9887 dB.

For example, three sources each reporting $L_{90} = 40$ dB give a logarithmic sum of 44.77 dB. Their sharp combined lower bound is 40 dB without independence and 43.01 dB with it. Independence therefore raises the bound by 3.0103 dB, numerically equal to the two-source error cap but describing a different comparison. If each also reports $L_{eq} = 46$ dB, the combined equivalent level is exactly $46 + 10 \log_{10} 3$ dB, approximately 50.77 dB. Equation (15) gives an unattained sharp upper endpoint of approximately 51.23 dB under arbitrary dependence. Independence preserves the 43.01 dB attained lower endpoint by Proposition 5.3; the upper bound need not then be sharp. Logarithmic addition of source percentiles is therefore unidentified. The exact sets behind equations (14) and (16), computable from the same reports, replace the point value; with two components, independence does not change them.

For fixed p , $\kappa_{m,p}/m \rightarrow 1-p$, so the equal-anchor error constant tends to $10 \log_{10}[1/(1-p)]$. Here $E[W] = (1-p) \sum_i \eta_i$ is the infimum of the combined mean energy by Proposition 3.3; for equal anchors, $e_p / E[W] \rightarrow 1$. At $p=0.10$ the limit is 0.4576 dB; equation (13) concerns a different percentile and target with the same numerical constant. Finite-source constants need not decrease (Figure 5).

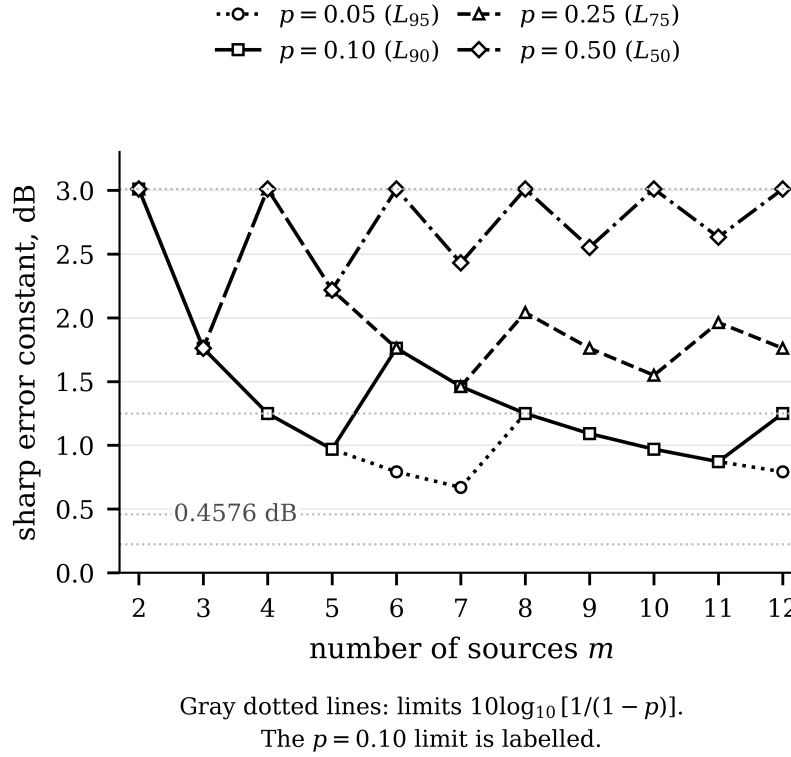


Figure 5. Sharp equal-anchor percentile-addition error under mutual independence, from equation (16), by source count and percentile rank. At $p=0.10$, $\kappa_{5,p}=\kappa_{6,p}=4$, so the error constant increases from 0.9691 dB to 1.7609 dB. Its limit is $10 \log_{10}(1/0.90)$.

Proposition 5.3 (Mean-Insensitivity). Adding prescribed finite means to Theorem 5.2 gives a nonempty class precisely when $\mu_i > (1-p)\eta_i$ for every i . For every such mean vector, the minimum remains e_p and is attained by bounded component laws.

Appendix C.2 constructs rare high-energy atoms that supply the prescribed means while preserving the combined quantile. No common upper support bound is imposed. Means can improve the upper endpoint but do not raise the sharp lower endpoint.

5.3. Subtracting percentiles and mixing descriptors

Percentile subtraction. Wind-farm assessment subtracts a background L_{90} from a total L_{90} . New Zealand wind-farm guidance, NZS 6808:2010, as quoted by Tonin [10], Sec. 1, calls this "not strictly mathematically correct" and permits it. The same standards body's general environmental-noise standard treats the removal differently: NZS 6802:2008 [27], B3.2.4 and B3.3.3, measures the residual sound level as an equivalent level and removes it by energy subtraction, and at B3.3.2 with Table B1 withholds assessment, absent special precautions, where the measured total sound level is 3 dB or less above the residual sound level. Let $x = q_X(p)$ and $\eta_Y = q_Y(p)$, the total percentile and the background percentile at the same rank, be the only reports, with $0 < \eta_Y \leq x < \infty$. Under nonnegative additive energies, the possible positive source anchors $\eta_S = q_S(p)$ form exactly $(0, x]$, with or without independence. For $x > \eta_Y$, logarithmic subtraction reports

$10 \log_{10}(x - \eta_Y)$. It can overestimate the source percentile level without limit; its greatest underestimation is $10 \log_{10}[x / (x - \eta_Y)]$ dB, attained at $\eta_S = x$. If $x = \eta_Y$, subtraction gives zero energy although every positive anchor up to x remains possible. Appendix C.1 proves these assertions. Percentile energies satisfy no subtraction identity analogous to that for means. For example, a total $L_{90} = 43$ dB and a background $L_{90} = 40$ dB give 39.98 dB by subtraction, whereas the source L_{90} can lie anywhere up to 43 dB and arbitrarily below.

Proposition 5.4 (Mixed-Descriptor Arithmetic). Let $0 < p < 1$ and $X = Y + S$, with $Y, S \geq 0$. Impose only the reports specified below. Under arbitrary dependence and under independence alike:

1. If $M = E[X] < \infty$ and $q_Y(p) = \eta_Y > 0$ are reported, the reports are compatible precisely when $M > (1 - p)\eta_Y$, and the exact source-mean set is

$$M_S \in [0, M - (1 - p)\eta_Y).$$

2. If instead $q_Y(p) = \eta_Y > 0$ and $0 < M_S = E[S] < \infty$ are reported, the possible finite total means and total percentile energies are, respectively,

$$M \in (M_S + (1 - p)\eta_Y, \infty), q_X(p) \in [\eta_Y, \infty).$$

Every included value is attained by bounded laws. Appendix C.2 gives the constructions. The source equivalent level is therefore strictly below $10 \log_{10}[M - (1 - p)\eta_Y]$. This upper bound is sharp, and there is no finite lower bound.

For source correction with $M > \eta_Y$, the computed level $10 \log_{10}(M - \eta_Y)$ can overestimate the source equivalent level without limit, even for positive source mean energy. Its underestimation has the sharp, unattained supremum

$$10 \log_{10} \frac{M - (1 - p)\eta_Y}{M - \eta_Y}.$$

For example, a total $L_{eq} = 50$ dB and background $L_{90} = 45$ dB give a corrected source level of approximately 48.35 dB. The sharp, excluded source-level ceiling is approximately 48.55 dB, 0.1963 dB above the correction; the source level can lie arbitrarily far below it. A nonpositive computed energy does not establish source inactivity.

ISO 1996-2:2017 [1], Sec. 10.4, Formula (16), prescribes energy subtraction when the residual level is more than 3 dB below the measured level. That separation condition does not itself justify substituting percentile energy for background mean energy.

Here the percentile describes the background during the assessed window. A source-off measurement requires justification that its percentile represents that background. A percentile taken from the total record, while the source is present, is a different input: Section 4.5 treats that substitution under its stated activity model.

For mixed addition, write $\Lambda = M_S + \eta_Y$. As an estimate of the combined equivalent level, $10 \log_{10} \Lambda$ overestimates, in decibels, by less than

$$10 \log_{10} \frac{\Lambda}{M_S + (1-p)\eta_Y} < 10 \log_{10} \frac{1}{1-p}.$$

The first bound is sharp but unattained. As in equation (15), it follows from the single-anchor inequality in Proposition 3.3. As an estimate of the combined L_N , its greatest overestimation is $10 \log_{10}(\Lambda / \eta_Y)$ dB, attained. Neither target has a finite bound on underestimation. A small one-sided bound is therefore not an accuracy guarantee.

For example, a background $L_{90} = 40$ dB and source $L_{eq} = 46$ dB give a mixed sum of approximately 46.97 dB. It overestimates the combined equivalent level by less than 0.0881 dB, but can overestimate the combined L_{90} by approximately 6.97 dB. A steady background and a source silent for at least 10% of the window attain the latter bound. A constant source energy instead makes the mixed sum exactly the combined percentile; its equivalent level alone does not establish constancy.

Neither percentile subtraction nor mixed-descriptor arithmetic generally identifies the stated target. The appropriate replacements are the compatible sets given above; an unsupported computed value need not belong to them.

5.4. Duty cycles and how often a combined level is exceeded

The remaining subsections turn from what the usual arithmetic cannot claim to what additional information can establish. Operating logs record how often each source runs. For intermittent sources, how often can the combined level reach a specified threshold, given equivalent levels and duty-cycle bounds? Independent sources taking energies zero or s satisfy $\Pr(X \geq s) = 1 - \prod_i (1 - \mu_i / s)$ when $\mu_i / s \leq \alpha_i$. The next result gives the sharp correction over an activity-bounded class; it does not assert sharpness at every fixed mean vector.

León and Perron [28], Eq. (6), establish binomial domination for convex objectives at a fixed total mean; the threshold objective is nonconvex. Strack and Westermann [29], Theorem 1, give the sharp bound

$$\Pr\left(\sum_{i=1}^m X_i \geq t\right) \leq 1 - (1 - 1/t)^m \text{ for independent nonnegative components with } E[X_i] \leq 1 \text{ and } t \geq 2m + 1.$$

Here individual activity bounds determine the correction.

Theorem 5.5 (Activity-Defect Reduction). Fix $\alpha = (\alpha_1, \dots, \alpha_m) \in [0, 1]^m$. Over mutually independent $U_i \in [0, 1]$ satisfying $\Pr(U_i > 0) \leq \alpha_i$, define

$$\Delta_m(\alpha) = \left[\Pr\left(\sum_i U_i \geq 1\right) - 1 + \prod_i (1 - E[U_i]) \right].$$

For independent $C_i \sim \text{Bern}(\alpha_i)$,

$$\Delta_m(\alpha) = \max_{z \in [0, 1]^m} \left[\Pr\left(\sum_i z_i C_i \geq 1\right) - 1 + \prod_i (1 - \alpha_i z_i) \right]. \quad (17)$$

Thus $U_i = z_i C_i$ attain the correction: z_i is the normalized energy while active. Equation (17) reduces probability-law optimization to amplitudes. Section S.5 of the Supplement proves the reduction and equations (19) and (20); Section S.1 of the Supplement extends the classification to unequal bounds.

For an energy threshold $s > 0$, set $U_i = \min\{X_i/s, 1\}$. This preserves the activity bounds and the event $X \geq s$. For finite means it gives the certificate

$$\Pr(X \geq s) \leq 1 - \prod_i (1 - \min\{\alpha_i, \mu_i/s\}) + \Delta_m(\alpha). \quad (18)$$

Here $\Pr(X \geq s)$ is the fraction of the window at or above level $10 \log_{10} s$; equation (18) bounds that fraction using equivalent levels and operating records.

For two sources with $0 < \alpha_i < 1$, put $\omega_i = (1 - \alpha_i)/\alpha_i$ and $\sigma = \sqrt{\omega_1} + \sqrt{\omega_2}$. Then

$$\chi_{act} = \max\left\{0, \max_{0 \leq z \leq 1} [z(1-z) - \omega_1(1-z) - \omega_2 z]\right\}, \Delta_2 = \alpha_1 \alpha_2 \chi_{act}. \quad (19)$$

The uncorrected product bound holds throughout the class if and only if $\sigma \geq 1$. For equal activity bounds, $\Delta_2(\alpha, \alpha) = \max\{0, 5\alpha^2/4 - \alpha\}$, with transition at $\alpha = 4/5$: the uncorrected product bound holds whenever each of two sources operates for at most 80% of the window. Without activity restrictions, the correction is $1/4$, attained by $U_1 = U_2 = 1/2$.

For three equal activity bounds, define

$$\alpha_a = \frac{15 - \sqrt{21}}{17}, \alpha_b = \frac{10 + 4\sqrt{2}}{17}, \alpha_c = \frac{99}{104}.$$

The exact correction is

$$\Delta_3(\alpha, \alpha, \alpha) = \begin{cases} 0, & 0 \leq \alpha \leq \alpha_a, \\ \frac{\alpha}{8}(-12 + 30\alpha - 17\alpha^2), & \alpha_a \leq \alpha \leq \alpha_b, \\ \frac{5}{4}\alpha^2 - \alpha, & \alpha_b \leq \alpha \leq \alpha_c, \\ -\alpha + \frac{\alpha^2}{3} + \frac{26\alpha^3}{27}, & \alpha_c \leq \alpha \leq 1. \end{cases} \quad (20)$$

Amplitude vectors $(1, 1, 1)$, $(1/2, 1/2, 1/2)$, $(1/2, 1/2, 0)$ and $(1/3, 1/3, 1/3)$ attain the respective branches of equation (20). For three equally bounded sources the transition is $\alpha_a \approx 0.6128$. At a 75% duty-cycle bound, every two-source correction vanishes, but $\Delta_3 = 45/512 \approx 0.0879$: a probability correction of about 8.79 percentage points is required. Checking every two-source subproblem therefore does not establish the many-source product bound. Section S.5 of the Supplement also proves that no positive common activity bound guarantees zero correction in every dimension.

5.5. Combined percentile ranges from equivalent levels, duty cycles and component bounds

When the correction vanishes, equivalent levels and activity bounds can give exact combined-percentile ranges. Theorem 5.6 concerns sources that can be silent; an always-present background fails its requirement $\alpha_i < 1$.

Theorem 5.6 (Combined Percentile Range for Intermittent Sources). Let $0 < \mu_i < \infty$, $0 < \alpha_i < 1$, and $\Delta_m(\alpha) = 0$. Consider mutually independent nonnegative components with means μ_i and $\Pr(X_i > 0) \leq \alpha_i$. For $0 < r < 1$, let $t_* > \max_i \mu_i$ be the unique root

$$\prod_i \left(1 - \frac{\mu_i}{t_*}\right) = r.$$

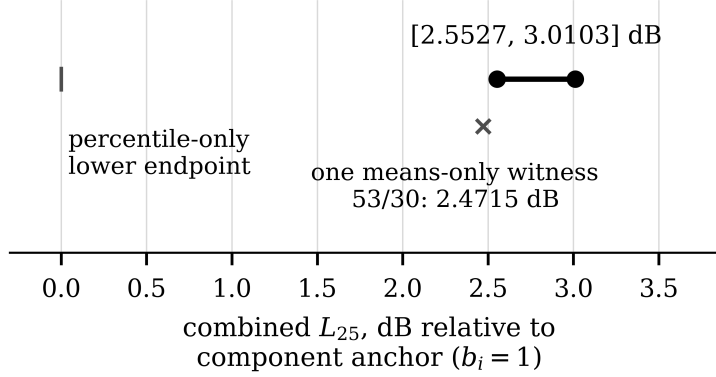
If $\mu_i / t_* < \alpha_i$ for every i , the exact combined-quantile set is $J_r = [0, t_*)$. Every value is attained by bounded component laws; the ceiling is excluded. Thus the combined L_N , with $r = 1 - N/100$, has ceiling $10 \log_{10} t_*$, calculated from the equivalent levels under the stated activity assumptions.

For any activity bounds, $r \leq \prod_i (1 - \alpha_i)$ instead forces $q_X(r) = 0$. In the regime of Theorem 5.6, the fixed-mean tail bound is also sharp:

$$\sup \Pr(X \geq s) = 1 - \prod_i \left(1 - \frac{\mu_i}{s}\right), \frac{\mu_i}{s} \leq \alpha_i. \quad (21)$$

Independent laws $X_i = s \text{ Bern}(\mu_i / s)$ attain equation (21). Section S.6 of the Supplement proves Theorem 5.6 and distinguishes this probability attainment from the excluded percentile ceiling. For two independent sources with equal equivalent levels and activity bounds 75%, the combined L_{90} ranges up to an excluded ceiling 1.6509 dB above either equivalent level, including $-\infty$ below (Figure 6).

(a) Percentiles and means



(b) Means and duty cycles $\alpha_i = 3/4$

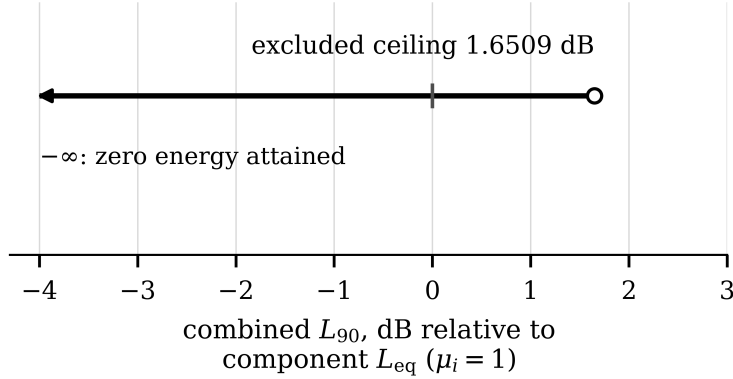


Figure 6. Identification intervals from additional component information. (a) Two sources on $[0, 1]$, with $p_1 = p_2 = r = 3/4$ and means $9/10$, give $[9/5, 2]$. The output is L_{25} , with levels relative to the component anchor $b_i = 1$. The percentile-only lower endpoint is 1. The cross, offset vertically below the interval for clarity, marks the feasible means-only value $53/30$, not a claimed endpoint; its horizontal position gives the level. (b) Means 1, activity bounds $3/4$ and $r = 0.10$ give $[0, (1 - 1/\sqrt{10})^{-1}]$. Here levels are relative to the component equivalent level. Filled and open endpoints indicate attainment and exclusion.

Theorem 5.7 (Joint Percentile-Mean Interval). Suppose mutually independent components satisfy $0 \leq a_i \leq X_i \leq b_i < \infty$, $a_i < b_i$, $q_{X_i}(p_i) = b_i$, and $\mu_i = b_i - d_i$ with $d_i > 0$ and $0 < p_i < 1$. Set $v_i = b_i - a_i$ and $b_\Sigma = \sum_i b_i$. Assume $\Delta_m(p_1, \dots, p_m) = 0$. For $0 < r < 1$, let $\tau > \max_i d_i$ be the unique root of

$$\prod_i \left(1 - \frac{d_i}{\tau}\right) = 1 - r.$$

If $\tau \leq \min_i v_i$ and $d_i/\tau < p_i$ for every i , then

$$I_r = \begin{cases} [b_\Sigma - \tau, b_\Sigma], & \text{some } v_i > \tau, \\ [b_\Sigma - \tau, b_\Sigma), & \text{every } v_i = \tau. \end{cases} \quad (22)$$

Here each reported percentile equals its component's maximum energy. Combining these anchors with equivalent levels restricts the combined percentile to equation (22), with both endpoint rules proved in Section S.6 of the Supplement. The deficit activity inequality $\Pr(X_i < b_i) \leq p_i$ follows from exact anchoring but is not equivalent to it. Interior anchors require separate treatment; Section S.4 of the Supplement gives a certified, potentially nonsharp two-source bound. For equal deficits, $\tau = d_1 / [1 - (1 - r)^{1/m}]$.

For two sources on $[0, 1]$ with $p_1 = p_2 = r = 3/4$ and $\mu_1 = \mu_2 = 9/10$, $\tau = 1/5$ and $I_r = [9/5, 2]$, corresponding to $[2.5527, 3.0103]$ dB. Without the means, independent Bernoulli($1/2$) components attain the sharp lower endpoint 1. Without the percentile constraints, take $X_1 = 9/10$ and let X_2 equal $13/15$ with probability $3/4$ and 1 otherwise. These means remain $9/10$, but the combined quantile is $53/30 < 9/5$. The improvement therefore follows from combining the information (Figure 6).

5.6. General reports: computable certified bounds

When several percentiles and an equivalent level are reported for each independent source, the combined range can be computed without the Gaussian analyses of [5] and [10], Sec. 2. Owhadi et al. [30], Theorem 4.1, give generalized-moment reduction for product measures; Pagès and Wilbertz [31], Sec. 2.3, develop conditional-mean-preserving splitting. Here exact anchors also require strict below-anchor inequalities.

Theorem 5.8 (Four-Seven Percentile-Energy Reduction). Suppose each nonnegative component has a prescribed finite mean and $k_i \geq 0$ exact lower-quantile anchors, and the independent compatible class is nonempty. The marginal constraints consist only of these reports and any declared support bounds. With or without finite upper support bounds, the infimum and supremum of $q_X(r)$ are unchanged when component i is restricted to at most $2k_i + 2$ atoms. Every value strictly between these endpoints is attained with at most $4k_i + 3$ atoms per component.

For one anchor the sufficient counts are four and seven atoms per source. Neither minimality nor endpoint attainment is asserted. Section S.7 of the Supplement proves the reduction.

For bounded supports, choose grids G_i containing every support endpoint and anchor value. Let $h_{G,i}$ be the largest grid gap, zero for a singleton, and $H_G = \sum_i h_{G,i}$. Write $\underline{q}(r), \bar{q}(r)$ for the unrestricted sharp endpoints, and $\underline{q}_G(r), \bar{q}_G(r)$ for the smallest and largest quantiles attained by exact grid laws. Then

$$\underline{q}_G(r) - H_G \leq \underline{q}(r) \leq \underline{q}_G(r), \bar{q}_G(r) \leq \bar{q}(r) \leq \bar{q}_G(r) + H_G. \quad (23)$$

Equation (23) holds at every output rank by mean- and anchor-preserving splitting (Section S.7 of the Supplement). Grid endpoints converge when $H_G \rightarrow 0$; nesting alone is insufficient. A level-error certificate requires a positive lower energy bound.

Section S.7 of the Supplement proves the reduction for anchors at any rank and for components without a declared upper support; Section S.4 gives the certified two-source interior-anchor bound. Section S.2 gives strict-boundary tests and tail constraints. In practice, the combined L_{90} of independent sources can be

certified from their reported percentiles, equivalent levels and support bounds, without fitting any distribution.

5.7. When sources are not independent

Sources sharing an operating schedule or a common driver, such as wind speed, need not be independent. Write d_{TV} for total-variation distance, the largest difference between probabilities assigned to the same event. The product law is the independence reference. A dependence budget must be externally justified or explicitly chosen for sensitivity analysis; component reports alone do not supply it.

Total variation alone bounds the distribution-function error by ε_{dep} . The next result sharpens the coefficient to $1 - 1/n_*$, determined by the smaller marginal support.

Theorem 5.9 (Staircase Dependence). Let P have two fixed finite marginals with support sizes n_1, n_2 , and let R be their product. Put $n_* = \min(n_1, n_2) \geq 2$. If $d_{TV}(P, R) \leq \varepsilon_{dep}$, then, with $\gamma_{dep} = (1 - 1/n_*)\varepsilon_{dep}$,

$$\sup_s |F_X^P(s) - F_X^R(s)| \leq \gamma_{dep}, q_X^R(r - \gamma_{dep}) \leq q_X^P(r) \leq q_X^R(r + \gamma_{dep}), \quad (24)$$

where the quantile bounds require $0 < r - \gamma_{dep} \leq r + \gamma_{dep} < 1$. For each n_* , the coefficient is sharp over the class of marginal pairs with that minimum support size; it need not be sharp for a specified pair. If $n_* = 1$, the coupling is unique.

The sum-threshold event forms a staircase-shaped subset of the two-component support grid; Section S.8 of the Supplement proves equation (24). For $n_* = 3$ and $\varepsilon_{dep} = 0.03$, the rank allowance is 0.02: the dependent combined L_{90} lies between the independent L_{92} and L_{88} . Their level separation depends on the distributions. Section S.3 of the Supplement gives exact finite transport extrema for specified marginals.

Two constructions delimit these guarantees (Propositions S.8.1 and S.8.2, Section S.8 of the Supplement). Pairwise independent components can preserve positive exact anchors, violate equation (16) and attain the arbitrary-dependence lower endpoint. Pairwise independent positive three-source laws can approach their product laws in total variation while the dependent law keeps combined median 4 and the product law 5. Small total variation alone therefore gives no uniformly vanishing decibel error. In Proposition S.8.2, a dependence that no pairwise check detects moves the combined median by $10 \log_{10}(5/4) = 0.9691$ dB. Ramachandra and Natarajan [32], Theorems 2.2 and 3.1, study probability bounds for pairwise independent Bernoulli sums; these constructions additionally address exact quantile constraints and quantile instability.

Table 3 separates what the reports establish about the two targets. Mean-energy sets list finite values; reports are positive, and only the stated constraints are imposed. Additional support or activity information can change these sets. For positive energies, logarithmic transformation preserves bounds and endpoint inclusion; zero corresponds to $-\infty$.

Table 3. What component reports identify about the combined sound. Arbitrary dependence unless mutual independence is stated.

Component reports and assumptions	Combined mean energy	Combined percentile energy $q_x(p)$
Means μ_i	$\sum_i \mu_i$, exactly	$[0, \sum_i \mu_i / (1-p))$
Common-rank percentiles η_i	$((1-p) \sum_i \eta_i, \infty)$	$[max_i \eta_i, \infty)$; error sign unidentified
Percentiles and compatible means	$\sum_i \mu_i$, exactly	$[max_i \eta_i, \sum_i \mu_i / (1-p))$, equation (15)
Source mean M_s and background percentile η_Y	$(M_s + (1-p)\eta_Y, \infty)$	$[\eta_Y, \infty)$ under either dependence assumption, Proposition 5.4
Percentiles; mutual independence	$((1-p) \sum_i \eta_i, \infty)$	$[e_p, \infty)$, equation (16); prescribing compatible means leaves the lower endpoint unchanged
Means and activity bounds, or upper anchors, means and supports; mutual independence	$\sum_i \mu_i$, exactly	Exact intervals under Theorems 5.6 and 5.7
Finite percentile packages and means; mutual independence	$\sum_i \mu_i$, exactly	Finite-atomic reduction; certified grid bounds on bounded supports
Fixed finite marginals and a total-variation dependence budget	$\sum_i \mu_i$, exactly	Rank bounds from equation (24), or exact finite transport optimization

For source extraction, the positive source percentile can lie anywhere in $(0, x]$ after percentile subtraction; correction of a total mean with a background percentile leaves the source mean in $[0, M - (1-p)\eta_Y)$ (Section 5.3).

5.8. Pooling percentile reports across time

Simultaneous sources add in energy; measurement windows pool as a duration-weighted mixture of distributions. Window equivalent levels and durations therefore identify the pooled mean energy exactly. Reports of a common percentile behave differently.

Proposition 5.10 (Pooled-percentile identification). Consider finitely many measurement windows with positive duration weights, for which only the same percentile L_N is reported in each window. Without further information about the within-window distributions, the percentile of the pooled record can take every value between the smallest and the largest reported window percentile. Both endpoints are attainable.

Write T_i for the window durations and $w_i = T_i / \sum_j T_j$ for the weights they give, and η_i for the energy anchor of the percentile window i reports. The exact set of pooled anchors is $[\eta_{min}, \eta_{max}]$, where

$\eta_{min} = \min_i \eta_i$ and $\eta_{max} = \max_i \eta_i$, and for positive anchors $10 \log_{10}$ carries that set to the interval of levels between the smallest and the largest reported L_N . Appendix C.4 proves the proposition; Appendix D gives a two-window witness.

For finite reported percentile levels, arithmetic averaging across measurement windows therefore does not in general identify the same percentile of the pooled record. The average is one point of the identified interval, and so are the smallest and the largest reported values. The replacement requires no additional measurement: its endpoints are already among the reported window percentiles.

5.9. Implications for measurement practice

Mean energies compose under addition and subtraction; percentile ranks leave unreported parts of the distribution free. A calculation targeting those parts generally returns a range. Its width is narrowed by named information, such as support, activity, exact recovery conditions, component means or dependence, and not by a universal correction factor. Percentiles remain valid descriptors and criteria in their own terms. An operation not analysed here requires its own identification argument.

Mixed correction illustrates why a small one-sided error allowance is insufficient. If total L_{eq} exceeds background L_{90} by $\Delta > 0$ dB, the sharp ceiling on the source level lies less than

$$10 \log_{10} \left[\frac{1 - 0.9 \cdot 10^{-\Delta/10}}{1 - 10^{-\Delta/10}} \right]$$

decibels above the usual correction: 0.1963 dB at $\Delta = 5$ dB. Yet the correction can overstate the source level without limit. If the total equivalent level is at or below background L_{90} but less than 0.4576 dB below it, the computed energy is nonpositive although positive source means remain compatible. Table 4 collects the sharp error statements for all operations treated here; Section 7 turns them into reporting instructions.

Table 4. Error of common percentile arithmetic. Error is the computed level minus the true level. Each bound is sharp over the information class of the cited result; "unbounded" means that no finite bound holds over that class. $\oplus$ and $\ominus$ denote addition and subtraction of levels in energy.

Practice	Estimates	Sharp overestimation bound	Sharp underestimation bound	Result
L_N reported as L_{eq}	L_{eq}	$< 10 \log_{10}(100/N)$: 0.4576 dB at L_{90} , 10 dB at L_{10}	unbounded	Proposition 3.3
L_{eq} taken as $L_{10} - 3$ dB [6]	L_{eq}	< 7 dB	unbounded	Proposition 3.3
Gaussian conversion, equation (8), without the Gaussian assumption	L_{eq}	at most the computed level minus $10 \log_{10} E_-$ from equation (5), using a declared lower support a ; see qualification below	unbounded without an upper support bound	Section 3.5
Source L_{eq} minus background L_N , read as a contrast with background L_{eq}	contrast	unbounded	$< 10 \log_{10}(100/N)$	Section 3.4
Total-record L_{95} as background, annex condition (quiet at least 5%), $\beta = 0$	background L_{eq}	unbounded	unbounded	Section 4.5, Appendix B.5
Total-record L_{95} as background, source active at most 5%	background L_{eq}	$< 10 \log_{10}(10/9)$ dB; unattained supremum approximately 0.4576 dB	unbounded	equation (13)
$L_{eq} \ominus$ background L_N	source L_{eq}	unbounded	for $M > \eta_Y$: $< 10 \log_{10} \frac{M - (1-p)\eta_Y}{M - \eta_Y}$; if $(1-p)\eta_Y < M \leq \eta_Y$, the computed energy is nonpositive although positive source means remain compatible	Proposition 5.4
Sum of component L_N values read as the combined L_{eq}	combined L_{eq}	$< 10 \log_{10}(100/N)$	unbounded	Table 3; Proposition 3.3

Practice	Estimates	Sharp overestimation bound	Sharp underestimation bound	Result
$L_N \oplus L_N$, m sources, any dependence	combined L_N	$10 \log_{10} \frac{\sum_i \eta_i}{\max_i \eta_i} \leq 10 \log_{10} m$	unbounded	equation (14)
$L_N \oplus L_N$, mutual independence	combined L_N	$10 \log_{10} \frac{\sum_i \eta_i}{e_p}$: 3.0103, 1.7609, 1.2494, 0.9691 dB for $m = 2$ to 5 equal L_{90}	unbounded	equation (16)
$L_N \ominus L_N$	source L_N	unbounded	for $x > \eta_Y$: $10 \log_{10} \frac{x}{x - \eta_Y}$; at $x = \eta_Y$, subtraction returns zero energy while positive source anchors in $(0, x]$ remain compatible	Section 5.3
Window L_{eq} values pooled by duration, read as the pooled L_{eq}	pooled L_{eq}	0	0	Section 5.8
Window L_N values averaged, read as the pooled L_N	pooled L_N	the average minus the smallest reported value	the largest reported value minus the average	Proposition 5.10
Background $L_N \oplus$ source L_{eq}	combined L_{eq}	$< 10 \log_{10} \frac{\Lambda}{M_S + (1-p)\eta_Y}$ $< 10 \log_{10} \frac{1}{1-p}$	unbounded	Section 5.3
Background $L_N \oplus$ source L_{eq}	combined L_N	$10 \log_{10} (\Lambda / \eta_Y)$	unbounded	Section 5.3
$L_{eq} \oplus L_{eq}$ and $L_{eq} \ominus L_{eq}$	combined or source L_{eq}	0	0	Section 5 opening

In the Gaussian-conversion row, use a declared lower support a , with $a=0$ available from nonnegativity alone. The bound is finite when $E_- > 0$ and is strict if any exact anchor exceeds a . This lower-energy bound does not require a finite upper support.

6. Consequences for standards and guidance

6.1. How these documents are read here

The documents below illustrate the operations proved in Sections 3 to 5; they are not an exhaustive survey. The results classify an operation wherever its actual inputs satisfy the theorem's information class. When they do not, the procedure must first be separated into its constituent steps. Omission of another document implies no judgement about it.

For source extraction, compatible total and residual equivalent levels subtract through their mean energies without a distributional model. The acoustic task is obtaining a residual mean that represents the assessed period. A background percentile supplies the range of Proposition 5.4 instead; a percentile of the total record invokes the distinct attribution problem of Section 4.5.

An energy identity, percentile criterion, rating index and model-based conversion can each be legitimate in its own terms. The assessment must preserve those terms. The proofs establish what the inputs identify; the prescribed procedure is then compared with that result.

6.2. Source correction and the meaning of residual sound

ISO 1996-2:2017 contains both cases within one document. Clause 10.4 applies logarithmic subtraction to measured and residual equivalent levels [1], Sec. 10.4. On the additive observation basis of Section 2.2 this is the equivalent-level form of the identity above, and it is exact for its stated target. The clause admits no correction when the residual level is 3 dB or less below the measured level. That restriction does not create the identity and is not required by it; what it restricts is a subtraction whose sensitivity to an error in either operand grows without bound as the two operands approach one another.

Annex I.2.1 is mathematically different. It proposes the total-record L_{95} as representing residual sound where the source does not contribute significantly during 5% or more of the measurement time [1]. The same substitution is named outside the annex: the note to the reporting requirement at Sec. 13 d) records that $L_{N,T}$ is "for example, used to estimate the residual sound using a typical value of $N = 95\%$ " [1]. This informative note records the substitution in the reporting clause; its location does not make the substitution a requirement or establish its accuracy. Section 4.5 tests that proposition. Even after strengthening "does not contribute significantly" to complete silence from the source, a source quiet for 5% of the window may be operating for the other 95%. Where no floor is claimed under the background, the levels consistent with the measurement include, for a finite reported L_{95} , a residual L_{eq} at least 13.0103 dB below that L_{95} , and they reach up to a residual L_{eq} as high as the total. The 0.4576 dB bound of equation (13) belongs to the opposite and far stronger condition, that the source is switched off for at least 95% of the window. Quiet for 5% of the time and switched off for 95% of the time are not the same condition, and the bound that holds under one does not hold under the other.

NZS 6802:2008 [27] separates the roles. Its Sec. 3 defines the background sound level as $L_{A90|t}$ and calls it the component of residual sound perceived as continuously present, but it is not one of the descriptors the standard's measurement and rating procedure or its noise limits are written in. The residual correction instead measures the residual sound level as an equivalent level and removes it from the measured equivalent level by energy subtraction (Secs. B3.2.4 and B3.3.3), and Sec. B3.3.2 with Table B1 withholds assessment, unless special precautions are taken and reported, where the measured total sound level is 3 dB or less above the residual sound level. Two editorial decisions recorded in the same document are of a piece with this: Commentary C5.3 records that the equivalent level replaced L_{10} as the standard's assessment descriptor in the 1999 edition, and the Foreword records that in the 2008 edition the term residual sound replaced the term background L_{eq} .

BS 4142:2014+A1:2019 [2] makes a further distinction. Ambient and residual sound levels are both defined as $L_{Aeq,T}$ (Secs. 3.3 and 3.10), and Sec. 7.3.3 corrects for residual sound on those quantities, normatively, by energy subtraction, again with a 3 dB condition diverting the user to another method. The background sound level is separately defined as $L_{A90,T}$, the level exceeded by the residual sound for 90% of the interval (Sec. 3.4), and Sec. 11 compares the rating level against it as a difference in decibels. These are two operations with different status. The first seeks a source energy and uses equivalent-level subtraction. The second is a defined assessment index, and a rating against background difference can be used in precisely those terms without asserting that L_{A90} is the background equivalent level. Difficulty arises only where the index is later reinterpreted as an energy decomposition it was not defined to provide. The separation was not always there. BS 4142:1990 [33] indexed its correction table on the noise level reading minus the background $L_{A90,T}$ (Table 1), a difference between an equivalent level and a percentile, and defined no residual level descriptor. That correction is the mixed-descriptor subtraction of Proposition 5.4, applied to recover a source equivalent level. The 1997 edition introduced residual noise level as an $L_{Aeq,T}$, moved the correction onto two equivalent levels and left the background percentile, in that correction, only as a permitted estimate of the residual, under stated conditions [34], 3.9, 3.10 and the note to Table 1. The current edition keeps that separation.

The 2026 United Kingdom wind turbine guidance [3] corrects for residual sound on percentiles, and it does so in two steps. Measurements are binned by wind speed, and Secs. 3.9 and 4.14 first replace the $L_{A90,10min}$ readings in each bin by their arithmetic average, separately for the total and the residual datasets. Secs. 3.9 and 4.20 then logarithmically subtract the residual bin value from the total bin value.

The two steps must be read separately. Proposition 5.10 governs the first: an arithmetic average of window percentiles does not identify the percentile of the pooled observations, which can be anywhere between the smallest and the largest reading in the bin. The bin average is therefore not established as the corresponding percentile of the pooled observations in that bin, and until it is, the sets of Section 5.3 cannot be attached to the subtraction that follows. Where the subtraction is performed on genuine percentile anchors, Section 5.3 gives what it establishes: subtracting one L_{90} from another can overstate the source L_{90} without limit, and its greatest understatement is the total L_{90} minus the level the subtraction returned, because the source L_{90} can be as high as the total L_{90} itself. At Secs. 3.9 and 4.21 the guidance adopts the same 3 dB margin for this correction that the three documents above adopt for their equivalent-level subtraction.

Secs. 2.46 and 2.48 apply the same arithmetic to the limits themselves. Sec. 2.46 derives a development's own limit by logarithmically subtracting the contributions of other wind turbine developments from the

total criterion, and Sec. 2.48 directs a check that the logarithmic addition of two developments' limits does not exceed that criterion. These are L_{A90} quantities, footnote 5 stating that every decibel value in the document is an L_{A90} unless explicitly stated. Those limits are regulatory quantities and are not percentiles of anything merely because they carry an L_{A90} label. The inference the check draws, however, is that developments individually meeting their L_{A90} limits will together meet the combined criterion, and that inference can be tested. Where the developments attain their limits on a common observation basis, the limits are the component anchors, and Proposition 5.1 applies to them: adding component L_{90} values in energy can overstate the combined L_{90} by at most 3.0103 dB for two sources and 4.7712 dB for three, and can understate it by any amount. Passing the addition check therefore does not establish that the combined L_{A90} meets the criterion.

The document performs that addition itself: footnote 10 adds the contributions of several developments each at a level 10 dB below the relevant noise limit, reports the result as exceeding that limit by about 1 dB "in theory", and judges the excess unlikely in practice and an acceptable marginal increase for cumulative operational wind turbine noise. Read as component anchors on a common basis, those limits do not bound the combined L_{A90} above at all [3], footnote 10.

Four documents, then, place a procedural threshold at the same 3 dB separation. In the three equivalent-level procedures, the underlying energy identity remains exact whatever the margin; the threshold governs whether or how the subtraction is used when the operands are close. In the percentile-based procedure, the same numerical separation creates no percentile identity.

6.3. Percentiles as criteria, and percentiles as inputs to other quantities

Where a criterion is expressed in a percentile, it can be assessed in that percentile without any equivalent level being estimated. The 2026 wind turbine guidance uses $L_{A90,10min}$ for background determination, for the setting of noise limits and for compliance measurement, seeking one index for all of those comparisons [3], Sec. 2.10. BS 4142 compares a rating level against a background $L_{A90,T}$ [2], Sec. 11. In that role the percentile is the quantity the criterion is written in, and Section 3.4 records the comparison as a valid index in its own terms.

At the other end, Ontario NPC-300 raises none of these questions: for the stationary-source background and limit procedure considered here it uses no percentile. The assessment descriptor is the one-hour equivalent level, the background sound level is expressed in the same quantity, and background enters as a floor beneath the applicable limit rather than as a quantity subtracted from a total [35], Secs. A5, B3, B4 and B5.

Between these lie the procedures that construct a criterion in one descriptor from a measurement in another. The NSW Noise Policy for Industry sets an intrusiveness criterion as the rating background noise level plus 5 dB, the left side being an $L_{Aeq,15min}$ and the right side derived from background L_{AF90} measurements [36], Sec. 2.3 and Fact Sheets A and B. For Quiet Areas, Irish EPA NG4 [4] derives the site criterion in $L_{Ar,T}$ or $L_{Aeq,T}$ by subtracting 10 dB from the average measured background level, where that average is formed arithmetically from the measured L_{AF90} values (Sec. 4.4.2). Its footnote vi marks the descriptor change explicitly: the measured background levels expressed as L_{AF90} are not to be confused with the licence-limit

values expressed as $L_{Ar,T}$ or $L_{Aeq,T}$. Read as written, each procedure defines a regulatory criterion from a background statistic; neither operation by itself asserts that a fixed offset converts a percentile into a mean energy. If such a criterion is later used as an estimate of a background or source equivalent level, that additional inference requires its own justification. Where the input is a genuine percentile of one record, Proposition 3.3 gives the corresponding identification result. Where the prescribed background statistic is itself constructed from percentile levels measured in separate windows, as in both procedures here, that construction must be analysed first: Proposition 5.10 applies to an arithmetic average, while any different aggregation rule requires its own argument. The resulting statistic cannot simply be treated as one percentile of one pooled record.

A fixed offset asserted between the two descriptors is a different matter. The 2026 guidance assumes, for prediction, that the L_{A90} lies 2 dB below the predicted L_{Aeq} [3], footnote 5. If the predicted L_{Aeq} and L_{A90} are interpreted as the mean level and the 90% exceedance percentile of one compatible nonnegative energy law, Section 3.4 permits the L_{A90} to lie anywhere below the L_{Aeq} plus the sharp 0.4576 dB ceiling, with no finite lower offset. The fixed 2 dB relation is therefore not identified by the two descriptors themselves; it remains the predictive convention adopted for the source class and conditions to which the guidance applies. The point is not particular to one document. Chiles records that the committee revising the New Zealand wind farm standard, facing the same problem with source data in L_{Aeq} and limits in L_{A90} , decided instead that a prediction based on L_{Aeq} source data is taken to be an L_{A90} , which he describes as providing a small degree of conservatism [37], p. 4. Two regimes, the same pair of descriptors, two different constants, each adopted as a decision.

Averaging percentile levels across windows appears in two of these documents. NG4 forms its background figure as the arithmetic average of the measured L_{AF90} values over the relevant period [4], Sec. 4.4.2, and the 2026 guidance takes an arithmetic average of the $L_{A90,10min}$ readings within each wind speed bin [3], Secs. 3.9 and 4.14. Under the NSW policy's long-term tenth-percentile method, the rating background level is the median across days of each period's tenth-percentile summary, subject to the policy's minimum rating background noise levels [36], Sec. 2.3, Table 2.1, and Fact Sheets A and B. The policy also permits directly measured period L_{AF90} values as assessment background levels (Fact Sheet B, Sec. B1.4). These are distinct procedures; neither is an arithmetic average of window percentiles. For the two arithmetic-averaging procedures, Proposition 5.10 gives what the averaging establishes: the percentile of the pooled record lies between the smallest and the largest reading averaged, and nothing in the readings says where. The average is one point of that interval, and so is the smallest reading, and so is the largest. The practical replacement costs nothing, because both endpoints are already in the survey.

These pooling operations belong to one comparison, and the documents supply both halves of it. ISO 1996-2 pools equivalent levels across sub-intervals as the duration-weighted energy mean, in its Sec. 10.1 d), Formula (15) [1]. That is the mean-energy composition of Section 5.8, and for mean energies it is exact, because a duration-weighted mixture has the duration-weighted average of the component mean energies. Its Sec. 6.1, Formula (5), then combines windows by their frequency of occurrence during the relevant period rather than by their measured duration, which is the distinction Section 8.5 draws between measurement composition and target composition, appearing here in the mean-energy setting. Annex I.2.1 of the same document replaces a residual mean energy by a percentile, and NG4 and the 2026 guidance pool percentiles by arithmetic average. Proposition 5.10 is what separates these from the first: pooling by duration identifies the pooled mean energy exactly from the window equivalent levels and their durations,

and identifies the pooled percentile only to the interval between the smallest and the largest reading. The point is not that one clause is sound and the others are not. It is that the composition rule which holds for a mean energy is written into the same document whose annex substitutes a percentile for such a mean. Nor would the documents that pool percentiles be repaired by adopting that rule: Proposition 5.10 holds for every set of positive duration weights, so no weighted combination of the window percentiles identifies the pooled percentile, and the choice between an arithmetic and an energy average is not what is at stake. Whether the windows measured represent a period that was not measured is a different question, and Section 8 takes it up.

Table 5. Assessment operations, the documents in which they appear, and what this paper establishes about them.

Operation	Document and clause	What this paper establishes	Reading
Total L_{eq} minus residual L_{eq} , in energy	[1] 10.4; [2] 7.3.3; [27] B3.3.3	Section 5 opening	Exact for total and residual mean energies of the same decomposition on a compatible basis; carrying a residual measured at another time or place into the assessed period is a representativeness question, not this one
A low percentile of the total record used as the residual L_{eq}	[1] Annex I.2.1	Section 4.5, equations (9) and (13), Appendix B.5	Not point-identified from the record. With an evidenced duty-cycle bound, report the residual-level interval Section 4 allows; without an informative activity restriction, no positive lower bound on the background follows in general
Percentiles converted to an L_{eq} by a Gaussian formula	[1] Annex I.2.2	Section 3.5, Table 4	Exact only where the level distribution really is the assumed one, which the report cannot show; otherwise not established
A rating level or criterion compared against a background percentile	[2] 11; [3] 2.10	Section 3.4	Valid as an index in its own terms; it is not the difference between two L_{eq} values and should not be read as one
A criterion set by a fixed decibel offset from a percentile or percentile-derived background statistic	[36] 2.3; [4] 4.4.2	Proposition 3.3 for a genuine single-record percentile; Proposition 5.10 or the prescribed aggregation rule first where the background statistic is built across windows	The offset may define the regulatory criterion. It does not by itself identify an equivalent background level; an aggregated background statistic must first be analysed according to the rule that constructs it.
A fixed decibel offset asserted between a percentile and an L_{eq}	[3] footnote 5; [37] p. 4	Proposition 3.3, Section 3.4	Allowed by the reports but not singled out by them; name the source class it was framed for and the use it is fit for
L_{A90} readings averaged within a bin, then one bin value subtracted from another	[3] 3.9, 4.14 and 4.20	Proposition 5.10, then Section 5.3 and Appendix C.1	Two operations. The average does not identify the bin percentile, so the bin value is not established as an anchor; on genuine anchors the subtraction can overstate the source without limit and understate it by as much as the total L_{90} minus the answer returned

Operation	Document and clause	What this paper establishes	Reading
Component L_{90} limits added in energy and compared with a combined criterion	[3] 2.46 and 2.48	Proposition 5.1, equation (14)	Where the developments attain their limits as common-rank anchors on a common basis, the sum overstates the combined L_{90} by at most 3.0103 dB for two and 4.7712 dB for three, and can understate it by any amount, so passing the check does not establish that the criterion is met
Background measured as an L_{eq} and used as a floor under the limit	[35] A5, B3, B4, B5	Not applicable	No percentile-to-energy identification question arises in this step; whether the monitored hour represents the operating period remains separate
Percentile levels averaged across windows	[4] 4.4.2; [3] 3.9 and 4.14	Proposition 5.10, Appendix C.4	The average does not identify the pooled percentile; report the range from the smallest reading to the largest, which needs nothing the survey does not already hold

6.4. What a measurement method should make explicit

For the practising acoustician, preserve a descriptor's meaning through the assessment. A background percentile measured while the source is off, a total-record percentile and a residual equivalent level are different inputs. Where the information determines a range, compute and report it; Section 7 gives the workflow.

For a standards committee, distinguish identities from conventions. A percentile criterion can remain in its own terms. A procedure seeking an equivalent source level should obtain compatible residual mean energy or state the assumptions identifying it. Gaussianity, fixed offsets, activity and dependence assumptions must remain visible. If only an interval is identified, retain it, require information that narrows it, or define an assessment index without claiming it is the unidentified physical quantity.

Representativeness remains separate. Exact arithmetic cannot rescue a residual measurement that does not represent the assessed period, and a representative survey does not make a percentile into mean energy. Use each descriptor for the quantity it identifies, and state the additional evidence whenever the assessment asks it to identify something else.

7. Measurement and reporting workflow

7.1. From identification result to measurement decision

Begin with the quantity the assessment requires. If it is the reported descriptor on the same observation basis, use that descriptor directly. If it requires conversion, subtraction, source combination or pooling, identify the information and assumptions needed for that operation before calculating.

The three classifications of Section 2.3 give three reporting instructions. An exact operation returns the target value. A valid index is reported in its defined terms. An unidentified operation is replaced by the compatible range, or by a narrower range after the additional information has been obtained. Choosing a single value within a range does not make it identified.

7.2. Information that must remain attached to the measurement

Retain the descriptor, frequency weighting, time weighting or integration interval, window duration and percentile rank. State whether percentiles came from sampled time-weighted levels or short-interval equivalent levels, and retain the averaging basis of any equivalent level used with them. Inputs must represent compatible energy observations and weights.

Also retain individual window values and durations when pooling; same-basis extrema when used as support bounds; resolution, rounding and class information; whether inputs came from the same or different windows; and the operating evidence and model assumptions used. These may include Gaussianity, mutual independence, activity restrictions, support bounds or a dependence budget. Align operating logs with the energy integration cells as required by Section 2.2. These items matter because the identification results depend on them, independently of whether a measurement standard also requires them.

7.3. Operations encountered in assessment

Table 6 follows the ten operations in Table 5. The observation and reporting information of Section 7.2 applies to every row. The table gives the additional inputs, the supported calculation and what an acoustician may report; the document clauses remain in Table 5.

Table 6. Measurement inputs, calculation and reporting for the operations of Table 5.

Assessment operation	Additional inputs and conditions	Calculation and result to report	Governing result
Total equivalent level minus residual equivalent level	Both equivalent levels must represent the same target decomposition. A separately measured residual must represent the assessed period.	Convert to energy, subtract, and convert the nonnegative result to level. This gives the source equivalent level exactly under those conditions; zero energy gives $-\infty$.	Section 5 opening; Section 8
Low total-record percentile used as residual equivalent level	Retain the full total record if available; otherwise retain its percentile package, total equivalent level, support bounds and reporting classes. Supply an evidenced activity bound.	For the full law, integrate the lowest-energy fraction of size $1 - \alpha$ and apply equation (9). For a package, use the joint percentile-and-mean refinement in equation (10). Report the background interval; report a point only when the inactive block, actual fraction and recovery conditions justify equation (11). Equation (13) is a separate sparse-activity bound.	Sections 4.1 to 4.5
Gaussian percentile-to-equivalent conversion	Percentiles on one compatible basis and an explicit Gaussian level model, supported by information beyond those percentiles.	Under that model, apply equation (8). Otherwise use equation (5) with justified support bounds; absent upper-tail control there is no finite upper endpoint. Report the model-based value or the supported interval accordingly.	Section 3.5; Theorem 3.1
Rating or criterion compared against background percentile	The quantities, time bases and rule defining the comparison.	Apply and report the defined index. It does not thereby become a contrast between equivalent levels.	Section 3.4
Criterion defined by a fixed offset from a background statistic	The statistic, offset and construction rule; retain individual values and durations if the statistic combines windows.	Construct the statistic and apply the prescribed offset. Report the criterion in those terms. A claim about pooled percentile or equivalent background level requires separate identification of that target.	Proposition 3.3; Proposition 5.10 or the applicable aggregation result

Assessment operation	Additional inputs and conditions	Calculation and result to report	Governing result
Fixed offset asserted between percentile and equivalent level	The two descriptors, source class and conditions within which the offset was adopted.	Report the declared convention within its scope. As a distribution-free relation on one compatible law, a finite reported L_N gives only $L_{eq} > L_N - 10 \log_{10}(100 / N)$, with no finite upper offset.	Proposition 3.3
Bin averages of window percentiles subsequently subtracted	Individual readings, durations and bin-statistic rules for both total and residual data.	Report a prescribed bin average as that statistic. For a pooled percentile, first report the smallest-to-largest window interval. Only genuine total and residual percentile anchors permit application of the source-percentile set in Section 5.3; the logarithmic subtraction is not an exact source percentile.	Proposition 5.10; Section 5.3
Component percentile values or limits added in energy	Genuine common-rank anchors on a common basis; establish this interpretation before treating regulatory limits as measurements. Retain any means, activity, support or dependence information needed for a stronger result.	Unknown dependence gives the lower endpoint equal to the largest component percentile. Justified independence gives the Bernoulli-envelope endpoint. Use the later theorems when their additional inputs and hypotheses hold. Report the resulting interval or bound; passing an arithmetic addition check alone does not establish compliance of the combined percentile.	Proposition 5.1; Theorems 5.2 and 5.5 to 5.9
Background equivalent level used as a floor under a limit	The measured background equivalent level and the rule defining the floor. Justify transfer if the measurement and target periods differ.	Apply the rule directly; for a rule taking the greater of a fixed criterion and background, take that greater value. Report the resulting criterion. No percentile conversion is required.	Section 8 for transfer
Percentiles averaged across windows	Each window value and duration, plus the rule defining the aggregate.	Report the arithmetic average if it is the required statistic. For the pooled percentile, report the interval between the smallest and largest readings, both included. Further within-window information is needed to narrow it.	Proposition 5.10

Three calculation details are particularly important. For the Gaussian model, equation (8) takes $L_{50} - L_{90}$, divides by $q_z(0.90) \approx 1.2816$, squares, multiplies by $\ln 10 / 20 \approx 0.1151$, and adds L_{50} . Use $q_z(0.95) \approx 1.6449$ with L_{95} . The rounded numbers explain the calculation; the exact constants define it.

For the independent percentile bound, convert each component level to energy. Enumerate all subset sums, weight each subset by the product of $1 - p$ over included components and p over excluded components, combine probabilities at equal sums, and sort. The first cumulative probability strictly greater than p selects e_p ; use $p = 0.10$ for L_{90} and 0.05 for L_{95} . Convert e_p to level and report it as an attained lower bound. This Bernoulli calculation computes the bound; it does not fit a source distribution.

For the dependence-budget branch, Theorem 5.9 requires two fixed finite marginal laws and an externally justified budget or an explicitly declared sensitivity choice. Read the independent distribution at the two shifted ranks of equation (24), checking that both lie in $(0, 1)$. The support counts are actual possible marginal values, not histogram bins. No corresponding budget result for more than two sources is proved here. Component reports establish neither independence nor a budget.

For genuine total and residual percentile levels $L_{tot} > L_{res}$, ordinary subtraction returns $10 \log_{10}(10^{L_{tot}/10} - 10^{L_{res}/10})$. The compatible positive source percentile can nevertheless be arbitrarily low and as high as L_{tot} . Equality of the two input levels makes the energy difference zero without excluding positive source percentiles. A bin average must first be established as a genuine anchor before this result applies.

The three background calculations likewise correspond to different information: equation (9) uses the full total law; equation (10) constrains a reduced report jointly with its mean; equation (11) uses identified inactive observations and recovery conditions. They are not interchangeable formulas for the same inputs.

7.4. How to report an identification set

Report the set as the range supported by the observations and declared assumptions. It is not an uncertainty band around a preferred estimate or a confidence interval. A record can determine such a range exactly.

Retain open endpoints and unbounded directions. An open endpoint can be approached but not reached by a compatible record; an unbounded direction must not be replaced by an arbitrary cap. Zero energy corresponds to $-\infty$, so a range reaching or approaching zero has no finite lower level endpoint.

A zero or negative energy produced by unsupported percentile subtraction does not establish source absence. Positive source energies may remain compatible. This differs from an exact subtraction of compatible means: a zero difference then identifies zero source mean, while a negative difference contradicts the stipulated nonnegative additive model.

7.5. Rounded and classed measurements

Retain displayed resolution, rounding convention, class intervals, whether percentiles were computed internally more finely than displayed, and class information for extrema used as bounds. A displayed percentile may represent an interval of underlying energies rather than one exact anchor.

The identification set is the union over all compatible exact-anchor vectors, keeping the total mean and every other constraint in each class. Substituting class limits independently is justified only when a separate argument proves equivalence to that union.

Finite reporting classes also do not establish finite underlying support. A histogram with a stated number of bins cannot, by itself, supply the support cardinality required by Theorem 5.9.

7.6. Reporting form

Table 7 provides a reporting form with seven entries that preserve the identification logic.

Table 7. Reporting form for an identification-based assessment.

Entry	What to record
Target quantity	The physical descriptor or assessment index required.
Observation basis	Weightings, integration or sampling basis, windows and durations.
Reported inputs	Values, ranks, extrema and reporting classes actually retained.
Operation classification	Exact, valid index, or unidentified.
Additional assumptions or external evidence	Only what the calculation uses: for example an operating log, support bound, Gaussian model, independence or dependence budget.
Identified result	The point value, defined index or complete supported range, including endpoint status and unbounded directions.
Representativeness beyond the window	The transfer claim and its evidence, if one is made; otherwise state that it has not been established.

7.7. From measured window to reference period

The workflow establishes what follows within the measured observation basis. Applying the result to another period, operating state or location requires the separate transfer justification in Section 8.

8. Representativeness beyond the measured record

8.1. Identification and external representativeness are separate questions

Identification asks what the measured record and its declared assumptions determine. Representativeness asks whether that quantity also applies to another period, place or operating condition. Neither answers the other. An exact within-record calculation can use an unrepresentative input, while a representative survey does not turn a percentile into a mean energy or establish source independence.

For example, compatible total and residual mean energies subtract exactly. The identity does not establish that a residual measured on another occasion equals the residual during the assessed period. This section makes that transfer statement explicit; it does not provide a general theory for validating it.

8.2. The quantity and domain being transferred

Specify both the target domain and the object transferred. Domains can differ in time, season, operating state, meteorology, location or regulatory reference period. The object may be a mean energy, percentile, distribution, activity fraction, support bound or dependence assumption.

Match the transfer claim to the calculation. A background percentile from another period does not supply the residual mean needed for energy subtraction; equal equivalent levels do not establish equal percentiles. Activity and dependence restrictions must apply to the target domain. Terms such as "background", "typical condition" and "representative measurement" leave these questions unresolved unless the required quantity and relation are named.

8.3. Residual sound measured in another window

When total and residual equivalent levels describe the same target decomposition on compatible bases, their nonnegative energy difference identifies the source mean without a distributional assumption. If the residual was measured in a separate window, the additional premise is that its mean applies during the assessed total-sound period. The arithmetic is exact conditional on that premise; greater numerical precision does not validate it. Keep the measured residual and the adopted target residual distinct in the report.

Selecting a quiet or source-off subset of the same total record is a different problem. Those observations already belong to the assessed window; the question is whether they identify its full-window background. Sections 4.3 to 4.4 address the inactive fraction, separation, selection error and possible background change between source-off and source-on periods. The eventual energy subtraction may look the same, but these two routes to its background input require different evidence.

8.4. The duration question in measurement language

"How long must I measure?" first requires an acoustic target: an hour, night, season or other reference period, and the descriptor sought. A window describes the observations it contains; a larger reference period also contains unobserved sound. The first question is what that window identifies about the reference-period descriptor.

Only if the remaining range can be made sufficiently narrow for a specified assessment tolerance can one ask for a smallest adequate duration. Neither existence of such a duration nor monotonic improvement with elapsed time is assumed. Section 8.9 states the further process assumptions needed to formulate this problem.

8.5. Several measured windows and an unmeasured reference period

Recorded durations determine the mixture of windows actually measured. Their equivalent levels identify its pooled equivalent level exactly; their same-rank percentiles identify its pooled percentile only within the range of Proposition 5.10.

Using that mixture for an unmeasured period adds a distinct claim. Measurement composition is the fraction of the survey spent in each window or condition. Target composition is the corresponding fraction in the assessed period. They need not agree. Pooling by observed duration identifies the observed mixture, not the composition of another period.

This distinction applies to operating modes, day and night, wind-speed classes, seasons and other observed conditions. Carrying a pooled result to a different composition requires the target proportions, or another explicit transfer assumption. Neither the mean-energy pooling identity nor the percentile interval alone supplies it.

8.6. Information and assumptions that do not automatically transfer

A record minimum or maximum bounds that record; it does not automatically bound a wider population or another period. The same applies to its activity and source-off fractions. Gaussianity, mutual independence and a dependence budget also belong to the conditions for which they are justified or chosen.

Keep these restrictions attached to the result and identify the domain to which each is asserted to apply. Information that narrows the measured-window interval cannot silently become evidence for the same restriction elsewhere.

8.7. Transfer across place

A background quantity measured at one position does not, by itself, identify the corresponding mean, percentile or distribution at another receiver. The spatial relation may be supplied by simultaneous observations, a measured comparison, a propagation model or another declared method. Its adequacy is external to the present identification results. Keep that transfer separate from the arithmetic performed after adopting a target residual.

8.8. Reporting a transfer claim

Every transfer report should retain the measured input, measurement period and location, target quantity and domain, transfer statement, supporting information or assumption, conditions observed, and adopted value or range. Table 8 identifies the issue for the principal cases.

Table 8. Transfers to be stated when a result is used beyond its measured domain. The common reporting fields above apply to every row; recording a transfer does not certify it.

Measured input and domain	Target and transfer being asserted	Supporting information and observed conditions	Result adopted
Residual L_{eq} in a separate window	Its mean applies to the assessed total-sound period for source extraction.	External basis for the transfer; operating and meteorological conditions during the residual measurement.	Target residual value or range.
Background L_{eq} in a monitoring period	It supplies the assessment-rule floor for the operating or regulatory period.	External basis and conditions covered by background monitoring.	Target background value or range.
Percentile or statistic from several windows	The measured composition applies to the reference period, or another target composition is supplied.	Composition evidence; conditions represented by each window and their durations.	Reference-period descriptor or statistic range.
Activity fraction from an operating record	The restriction applies during the target period.	Operating information or declared assumption; operating states covered.	Activity bound or range.

Measured input and domain	Target and transfer being asserted	Supporting information and observed conditions	Result adopted
Record extrema or other support information	The bound applies to the target period or population.	External support justification; conditions in which the extremes occurred.	Support interval.
Dependence assumption under characterized conditions	It applies under the target operating conditions.	External justification or sensitivity assumption; conditions covered by component characterization.	Resulting identified range.
Residual measured at another position	A spatial relation carries the required quantity to the assessment receiver.	Declared spatial relation or model; measurement and propagation conditions.	Target-receiver value or range.

Claims that a transfer establishes stability, that a duration is adequate or that a measurement is representative require evidence beyond these within-record results.

8.9. The open reference-period identification problem

An unconstrained remainder need not make every reference-period descriptor unidentified. If 96% of a reference period is observed at one constant positive energy, the remaining 4% can place at most 4% probability below it, while the cumulative probability at it is at least 96%. The reference-period L_{95} is fixed, although the equivalent level is not. If instead the remainder is required to have the same distribution as the observed window, both descriptors are identified by that common distribution.

Between these extremes, let C be a declared class of admissible acoustic processes, θ the target of Section 2.3, R_T the observed record of duration T , and $T_R > T$ the required reference period. The object sought is

$$\left\{ \theta(P_{R_{T_R}}) : R_{T_R} \text{ extends } R_T \text{ and is admissible under } C \right\},$$

where $P_{R_{T_R}}$ is the law of the reference-period extension. Which process classes restrict this set usefully? How does its width change with observation duration? Does a duration exist that brings it below an assessment tolerance, and if so what is the smallest one?

The answers can differ between equivalent levels and percentiles: unobserved energy changes a mean, whereas the placement of unobserved observations changes a percentile rank. The continuation classes, sampling theory and long-term data needed to answer these questions are not developed here. The paper therefore supplies no universal minimum duration, stationarity test, seasonal correction or general representativeness criterion. It identifies the within-record result first and makes the additional transfer premise visible.

9. Conclusions and future work

9.1. What the reported descriptors identify

Recovering background mean energy from a measured total requires more than selecting L_{90} or L_{95} . Those percentiles report ranks, whereas the background equivalent level depends on its full-window mean energy,

including periods when the source operates. Their common decibel unit does not make these quantities interchangeable.

The replacement for an unsupported point value is a sharp compatible range. Percentiles with justified support bounds give the mean-energy interval of Theorem 3.1. A total distribution with evidenced activity gives the background and source intervals of equation (9); a reduced report with its total mean gives equation (10). Identified inactive observations and the recovery conditions of Theorem 4.2 can turn the background range into a point. Theorem 4.3 distinguishes selecting the wrong observations from a change in background during source operation.

The activity assumption matters quantitatively. Exact inactivity during 5% of a window and exact inactivity during 95% are different information. Section 4.5 shows why the former does not justify a universal total-record L_{95} substitution; the latter gives the sharp, unattained one-sided supremum $10 \log_{10}(10/9) \approx 0.4576$ dB, with no finite opposite-direction bound over the stated class. Background floors and support restrictions must remain explicit.

9.2. Combining sources and combining measurement windows

Compatible mean energies add and subtract exactly. Component percentile energies do not. The paper gives replacement sets under unknown dependence, a sharp independent-source envelope, and refinements using means, duty cycles and support information. Prescribing means alone does not raise the independent anchor-only lower endpoint. The exact product-root intervals require their stated zero-defect and strict activity conditions; they are not general formulas for every source class.

For independent finite percentile reports and means, Theorem 5.8 supplies sufficient finite-atomic counts, and equation (23) supplies certified bounded-support grids when the largest gap tends to zero. For two fixed finite marginals, Theorem 5.9 bounds the effect of a declared dependence budget; Supplement S.3 gives the exact fixed-marginal optimization. Small probability perturbations alone need not give small level errors at a percentile.

Pooling windows is a duration-weighted mixture. Window equivalent levels and durations identify the pooled equivalent level, whereas window percentiles alone permit the entire interval between their smallest and largest values. The arithmetic average is a possible value, not the identified pooled percentile.

9.3. Consequences for measurement and standards

For an acoustician, the task is to preserve each descriptor's meaning through the assessment. A percentile criterion remains valid in its own terms. When a source or residual equivalent level is required, obtain compatible mean-energy information or state the assumptions supporting its recovery. When the inputs support a range, report that range or obtain the information that narrows it.

Standards and guidance should distinguish identities, model-based conversions, assessment indices and physical quantities not identified by their inputs. A prescribed convention can define an index without establishing that it equals a different acoustic target. Section 7 gives the practical workflow: retain the observation basis and evidence, classify the operation, and report the supported result with its open endpoints and unbounded directions. An unsupported percentile subtraction returning zero or negative energy does not establish source absence.

9.4. Scope and remaining mathematical limits

These are structural results for a population law or complete observation window, not finite-sample confidence intervals. Sampling inference requires its own design and assumptions. Extrema, activity fractions and dependence information apply to the domain for which they are justified; component reports do not establish independence.

The mathematical limits remain explicit. Theorem 5.8 gives sufficient atom counts without claiming minimality or general endpoint attainment. Theorem 5.7 requires upper-support anchors; the interior-anchor result in Supplement S.4 is a certified, potentially nonsharp two-source bound. Theorem 5.9 concerns two fixed finite marginals, with a coefficient sharp over the support-size class rather than necessarily for each specified pair. The activity-defect reduction applies to arbitrary source count, but closed forms are established only in the stated cases. Checking all two-source subproblems does not establish a many-source product bound.

9.5. Future work: from a measured window to a reference period

Section 8 formulates the next measurement question: what does the observed record establish about a specified descriptor over an incompletely observed reference period? A duration criterion can follow only after defining the target, admissible processes and acceptable remaining range. Equivalent levels and percentiles need not require the same observation duration.

Three routes should remain distinct. A declared process class can constrain the unobserved continuation through activity, diurnal structure, operating regime or stationarity. A sampling design, such as randomized sub-windows, can support estimation with a confidence level. Long-term reference data can test or calibrate the process class used for transfer. Exposure or annoyance applications likewise require the descriptor to be identified or statistically estimated over the period to which the response relation applies.

Further mathematical work concerns exact interior-anchor results, dependence budgets beyond two fixed finite marginals, sharper bounds for specified marginals, minimal finite-support representations, endpoint attainment and additional activity-defect closed forms. The practical question remains specific: for the acoustic target and observation design at hand, what does the measurement support, and what further observation or evidence is needed to make that result useful for assessment?

Timestamped measurement records voluntarily contributed through the application at <https://www.noisepercentiles.com> will support a companion study of measurement duration and reference-period representativeness.

Data availability statement

No experimental measurement dataset was collected or analysed in this theoretical study. All synthetic distributions and numerical inputs underlying the examples are specified in the article, its appendices and the accompanying Supplement. The Technical Mathematical Supplement, the figure-code package containing the Python source, dependency specification, reproduction instructions and verification output, and the generated Figures 1 to 6 in PDF, PNG, EPS and JPG formats are deposited at Zenodo DOI [10.5281/zenodo.23088962](https://doi.org/10.5281/zenodo.23088962). A browser application for evaluating the identification sets developed in Sections 3 to 5 from measured records or reported values, together with its source code and self-tests, is available at <https://www.noisepercentiles.com>.

Supplementary Material

The Technical Mathematical Supplement contains the unequal-activity classification, finite boundary and tail tests, fixed-marginal transport extrema, interior-anchor bounds, the complete specialist proofs of Theorems 5.5 to 5.9, and the consolidated notation register. The accompanying figure-code package provides the program and instructions for reproducing Figures 1 to 6; the generated figures are also supplied. These materials are separate from the printed article and will be accessible through the Zenodo deposit identified in the Data availability statement.

Use of AI tools

All mathematical derivations in this paper were carried out by hand. ChatGPT (OpenAI, SOL 5) was used to assist with literature searches, the identification and checking of relevant prior work and the development of Python plotting code for Figures 1 to 6 and code for the browser application described in the Data availability statement. The figures are deterministic outputs obtained by executing plotting code using the mathematical formulas and numerical inputs specified in the paper; no generative image tools were used. The author takes full responsibility for the paper's content, including the mathematical results, interpretation of the literature, text, code and figures.

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Any remaining errors are my own.

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Appendices

Appendix A. Proofs for Section 3: quantile representation and energy bounds

A.1. Quantile representation

We use the generalized inverses defined in equation (1). Lemma A.1 records the inverse relation, representation and converse needed for the constructions in Section 3.

Lemma A.1 (Quantile representation and converse). Let Z be a real-valued random variable. Its lower quantile is finite, nondecreasing, and left-continuous on $(0, 1)$, and

$$q_Z(u) \leq x \Leftrightarrow u \leq F_Z(x), 0 < u < 1. \quad (\text{A.1})$$

If U is uniform on $(0, 1)$, then $q_Z(U)$ has the distribution of Z . Conversely, every finite-valued, nondecreasing, left-continuous function $\tilde{q}$ on $(0, 1)$ is the lower quantile of $\tilde{q}(U)$. The lower and upper quantiles differ at no more than countably many ranks.

Proof. Since F_Z tends to zero at negative infinity and to one at positive infinity, $q_Z(u)$ is finite for every $0 < u < 1$. Right continuity of F_Z gives $F_Z(q_Z(u)) \geq u$. Monotonicity of F_Z then proves equation (A.1). Monotonicity of q_Z follows from the nesting of the sets in its definition.

For left continuity, suppose that $\sup_{0 < s < u} q_Z(s) < q_Z(u)$. Choose x strictly between these values. Equation (A.1) gives $F_Z(x) \geq s$ for every $s < u$, hence $F_Z(x) \geq u$. A second application of equation (A.1) contradicts $x < q_Z(u)$. Thus q_Z is left-continuous. Moreover,

$$\Pr(q_Z(U) \leq x) = \Pr(U \leq F_Z(x)) = F_Z(x),$$

which proves the representation.

For the converse, monotonicity gives

$$\Pr(\tilde{q}(U) \leq \tilde{q}(u)) \geq u.$$

If $x < \tilde{q}(u)$, left continuity supplies $s < u$ with $\tilde{q}(s) > x$. Monotonicity then gives

$$\Pr(\tilde{q}(U) \leq x) \leq s < u.$$

These two inequalities identify $\tilde{q}(u)$ as the lower quantile at rank u .

Finally, $q_Z(u) \leq q_Z^+(u)$. For $0 < u < s < 1$, the inequality $F_Z(q_Z(s)) \geq s > u$ implies $q_Z^+(u) \leq q_Z(s)$. Consequently, the nonempty intervals $(q_Z(u), q_Z^+(u))$ are pairwise disjoint. Each contains a rational number, so there are at most countably many such intervals. $\square$

For nonnegative energy, equality in distribution yields

$$E[X] = E[q(U)] = \int_0^1 q(u) du,$$

including an infinite value. The strictly increasing transformation $X = 10^{L/10}$ gives $F_X(10^{z/10}) = F_L(z)$ for finite z . With $10^{-\infty} = 0$, the exponential map is a continuous increasing bijection from $[-\infty, \infty)$ to $[0, \infty)$. It carries the sets defining the generalized inverses to one another and preserves their infima. Hence $q(u) = 10^{Q(u)/10}$, including ranks corresponding to a zero-energy atom. These identities prove equation (2) under its stated measurement-basis assumption.

The exact-anchor criterion in Section 2.1 also follows from equation (A.1): $q_Z(u) = z$ if and only if $F_Z(z) \geq u$ and $F_Z(x) < u$ for every $x < z$. Taking the left limit of the latter inequalities gives

$$\Pr(Z < q_Z(u)) \leq u \leq \Pr(Z \leq q_Z(u)).$$

Setting $u = 1 - N/100$ and taking complementary probabilities yields

$$\Pr(L > L_N) \leq N/100 \leq \Pr(L \geq L_N).$$

The bracket relation alone does not impose the strict inequalities required at every point below an exact anchor.

A.2. Sharp retained-integral bounds and endpoint attainment

Fix an integer $k \geq 0$ and support bounds $0 \leq a \leq b < \infty$. If $k \geq 1$, let $0 < u_1 < \dots < u_k < 1$ and $a \leq q_1 \leq \dots \leq q_k \leq b$ be the reported ranks and energy anchors. For $k=0$, the package is empty. The compatible class consists of all laws supported on $[a, b]$ with $q_X(u_j) = q_j$; no further constraints are imposed. As in Section 3, set $u_0 = 0$, $u_{k+1} = 1$, $q_0 = a$, and $q_{k+1} = b$.

Proposition A.2 (Sharp retained-integral bounds). For every $u \in [0, 1]$, the attainable values of $\int_0^u q_X(s) ds$ are exactly

$$\begin{cases} [\underline{E}_u, \bar{E}_u], & q_j = a \text{ for every } j \text{ with } u_j \leq u, \\ (\underline{E}_u, \bar{E}_u], & \text{otherwise,} \end{cases} \quad (\text{A.2})$$

where the bounds are the finite sums in equation (6). The condition in the first line is vacuous if no reported rank lies in $(0, u]$.

Proof. For $k=0$, the bounds are $\underline{E}_u = au$ and $\bar{E}_u = bu$. Constant quantiles with values in $[a, b]$ attain every value in $[au, bu]$, which proves the assertion in this case. Assume henceforth that $k \geq 1$.

On each open interval (u_j, u_{j+1}) , monotonicity and the support bounds give

$$l(s) = q_j \leq q_X(s) \leq q_{j+1} = h(s).$$

Integration proves both inequalities in equation (6). Values at the finitely many interval boundaries do not affect these integrals.

For upper attainment, choose the left-continuous representative

$$h(s) = q_{j+1}, s \in (u_j, u_{j+1}], 0 < s < 1.$$

It satisfies $h(u_j) = q_j$ at every reported rank. Lemma A.1 therefore makes $h(U)$ a compatible energy variable with lower quantile h . Its retained integral equals $\bar{E}_u$ for every u . Explicitly, this law assigns mass Δu_j to energy q_{j+1} , combining masses when consecutive energies coincide.

For the lower bound, give l the values q_j on $(u_j, u_{j+1}]$, restricted to $(0, 1)$. Choose

$$0 < \varepsilon < \min_{0 \leq j < k} \Delta u_j$$

and define

$$q_\varepsilon(s) = \begin{cases} q_j, & u_j - \varepsilon < s \leq u_j \text{ for some } 1 \leq j \leq k, \\ l(s), & \text{otherwise.} \end{cases}$$

The modified intervals are disjoint and lie immediately before their respective anchors. At their left endpoints, the value remains the preceding lower value. Thus q_ε is nondecreasing and left-continuous, takes values in $[a, b]$, and satisfies every exact anchor. By Lemma A.1 it is an admissible quantile. Furthermore,

$$0 \leq \int_0^u (q_\varepsilon(s) - l(s)) ds \leq k\varepsilon(b - a).$$

Letting ε decrease to zero proves that $\underline{E}_u$ is the sharp infimum.

Suppose an anchor at or below u exceeds a , and let i be the first such index. Then $q_{i-1} = a < q_i$. Left continuity of any compatible quantile forces its values to exceed a by a fixed positive amount on some interval immediately preceding u_i . On that interval, $l = a$. Since $q_X \geq l$ almost everywhere, the retained integral is strictly greater than $\underline{E}_u$. The lower endpoint is therefore excluded.

Conversely, suppose every anchor at or below u equals a . In the construction of q_ε , choose ε smaller than every positive difference $u_j - u$ for anchors above u . All modifications that raise the value above a then lie strictly above u . Consequently, $q_\varepsilon = l = a$ almost everywhere on $(0, u)$, and the lower endpoint is attained. If there are no anchors above u , the constant quantile a suffices. At $u = 0$, every retained integral is zero.

It remains to exclude gaps between attainable values. Let $q^{(0)}$ and $q^{(1)}$ be two compatible quantiles. For $0 \leq t \leq 1$, define their pointwise interpolation by

$$q^{(t)}(s) = (1-t)q^{(0)}(s) + tq^{(1)}(s).$$

This function preserves monotonicity, left continuity, support, and every exact anchor. Lemma A.1 makes it a compatible quantile, and

$$\int_0^u q^{(t)}(s) ds = (1-t) \int_0^u q^{(0)}(s) ds + t \int_0^u q^{(1)}(s) ds.$$

Every value strictly between the sharp bounds lies between two attainable values and is therefore attained by such an interpolation. Together with the endpoint constructions, this proves equation (A.2). $\square$

Taking $u = 1$ in Proposition A.2 and applying equation (2) proves Theorem 3.1, including its endpoint rule. The strictly increasing map $M \mapsto 10 \log_{10} M$ carries the mean-energy set to the equivalent-level set and preserves endpoint inclusion, with $10 \log_{10} 0 = -\infty$. Subtracting a fixed finite prediction from every compatible level translates this set and preserves its endpoint inclusion. If $a = b$, both identification sets are singletons.

Proof of Proposition 3.3. For the single positive anchor $q(p)$ without an upper support bound, monotonicity and left continuity give $\int_0^1 q(u) du > (1-p)q(p)$: the integral above p is at least $(1-p)q(p)$, and the quantile is positive on an interval immediately before p . Conversely, on $[0, b]$ with $b \geq q(p)$, the one-anchor case just proved attains every finite mean in $((1-p)q(p), pq(p) + (1-p)b]$. Letting $b \rightarrow \infty$ gives exactly $((1-p)q(p), \infty)$ for finite means, and logarithmic transformation gives the stated level inequality. Every such mean is attained by a bounded law. $\square$

A.3. Monotonicity, refinement, and rank selection

For fixed ranks and support bounds, each anchor coefficient in equation (6) is nonnegative. Increasing any anchors while preserving their order therefore cannot decrease either step bound. This proves Lemma 3.2 for every retained rank, including zero and one.

Now insert an exact anchor of value z at a rank $s \in (u_j, u_{j+1})$, where $q_j \leq z \leq q_{j+1}$. The lower step value remains q_j before s and becomes z after s . The upper step value becomes z before s and remains q_{j+1} after s . Thus refinement raises l and lowers h almost everywhere. Integration proves that every lower retained-integral bound increases or remains unchanged, while every upper bound decreases or remains unchanged. When $E_- > 0$, the same ordering implies that $10 \log_{10}(E_+ / E_-)$ cannot increase.

For the width, nonnegativity of $h - l$ and telescoping of the anchor increments give

$$0 \leq \bar{E}_u - \underline{E}_u \leq E_+ - E_- = \sum_{j=0}^k \Delta u_j (q_{j+1} - q_j) \leq \Delta_{\max} (b - a). \quad (\text{A.3})$$

Choose an index i with $\Delta u_i = \Delta_{\max}$. Set $q_j = a$ for $1 \leq j \leq i$ and $q_j = b$ for $i < j \leq k$. This feasible package places the entire increment $b - a$ in the i th gap and attains equality in the full-width bound. Hence the worst-case width in equation (7) is exactly $\Delta_{\max} (b - a)$.

The $k + 1$ positive gaps sum to one. Their maximum is therefore at least $1 / (k + 1)$, with equality only when every gap equals $1 / (k + 1)$. If $a < b$, the unique minimizing ranks are consequently $u_j = j / (k + 1)$. If $a = b$, every rank choice gives zero width. For $a < b$, every fixed finite package has positive width because at least one anchor increment is positive and every rank gap is positive.

Equation (A.3) also proves convergence uniformly in the retained rank whenever the packages describe one law on a common bounded support and $\Delta_{\max} \rightarrow 0$. Both bounds converge to its retained integral. For a

positive mean, continuity of the logarithm gives convergence of the equivalent-level bounds as well. A varying support width is not covered by this conclusion.

A.4. Gaussian percentile conversions

Assume the level distribution is Gaussian with mean m_L and standard deviation $s_L \geq 0$. Represent it as $L = m_L + s_L Z$, where Z is standard normal. Completing the square gives, for every real x ,

$$\begin{aligned} E[e^{xZ}] &= \frac{\int_{-\infty}^{\infty} e^{-z^2/2+xz} dz}{\int_{-\infty}^{\infty} e^{-z^2/2} dz} \\ &= e^{x^2/2} \frac{\int_{-\infty}^{\infty} e^{-(z-x)^2/2} dz}{\int_{-\infty}^{\infty} e^{-z^2/2} dz} = e^{x^2/2}. \end{aligned}$$

It follows that

$$E[10^{L/10}] = \exp\left(\frac{\ln 10}{10} m_L + \frac{(\ln 10)^2}{200} s_L^2\right), L_{eq} = m_L + \frac{\ln 10}{20} s_L^2.$$

For $s_L > 0$, the increasing affine transformation of Z gives

$$L_N = m_L + s_L q_Z(1 - N/100), L_{50} = m_L.$$

Normal symmetry gives $q_Z(1 - N/100) = -q_Z(N/100)$. Therefore, for $N \in \{90, 95\}$,

$$s_L = \frac{L_{50} - L_N}{q_Z(N/100)}.$$

Substitution into the equivalent-level identity proves equation (8). For the symmetric percentile pair,

$$m_L = \frac{L_{10} + L_{90}}{2}, s_L = \frac{L_{10} - L_{90}}{2 q_Z(0.90)},$$

and hence

$$L_{eq} = \frac{L_{10} + L_{90}}{2} + \frac{\ln 10}{80 [q_Z(0.90)]^2} (L_{10} - L_{90})^2.$$

If $s_L = 0$, every percentile equals m_L , and the same identities hold. For $s_L > 0$, the Gaussian level and its lognormal energy have unbounded upper support. The Gaussian assumption therefore defines a different information class from the bounded-support class of Theorem 3.1. The identities determine the equivalent level within that Gaussian class; two reported percentiles alone do not establish membership in it.

Appendix B. Proofs for Section 4: background attribution and recovery

Appendix B.1 proves equation (9) and the compatibility criterion of Section 4.1; Appendix B.2 proves Theorem 4.1; Appendices B.3 and B.4 prove Theorems 4.2 and 4.3; Appendix B.5 proves the percentile bracket of Section 4.4 and the statements of Section 4.5; Appendix B.6 treats indivisible cells.

B.1. Selection and compatible background laws

Throughout this appendix, $X = Y + S$, $Y \geq \beta \geq 0$, $S \geq 0$, and $M = E[X] < \infty$. Unless further restrictions are stated, the admissible class consists of all joint laws with the reported total-energy information and $Pr(S > 0) \leq \alpha$, where $0 \leq \alpha \leq 1$ and $c = 1 - \alpha$. Probability spaces may be enlarged to divide mass at a threshold atom. Indivisible observation cells are treated separately in Appendix B.6.

The continuous-law threshold formulation of tail-mean optimization is given by Rockafellar and Uryasev [21], Sec. 2, Theorem 1. The selection argument needed here includes atoms.

Lemma B.1 (Optimal Tail-Selection). For a nonnegative integrable X and $0 \leq c \leq 1$, there is an event B of probability c , on an enlarged probability space if necessary, such that

$$E[X 1_B] = J_c, E[X 1_E] \geq J_c \text{ whenever } Pr(E) \geq c. \quad (\text{B.1})$$

If $0 < c < 1$ and $Pr(E) = c$, equality holds precisely when E contains $\{X < q(c)\}$ and excludes $\{X > q(c)\}$, up to null sets.

Proof. Suppose $0 < c < 1$ and set $z = q(c)$. The quantile inequalities give $Pr(X < z) \leq c \leq Pr(X \leq z)$. Select all of $\{X < z\}$ and the required fraction of $\{X = z\}$, using an independent uniform variable on that atom. This defines B . By equation (A.1), $q(u) < z$ if and only if $u \leq F_X(x)$ for some $x < z$. Thus $\{q(U) < z\}$ and $\{U < Pr(X < z)\}$ differ by a null set, where U is uniform on $(0, 1)$, and quantile representation gives

$$\begin{aligned} E[X 1_B] &= E[X 1_{\{X < z\}}] + z[c - Pr(X < z)] \\ &= \int_0^{Pr(X < z)} q(u) du + \int_{Pr(X < z)}^c q(u) du = J_c, \end{aligned}$$

because $q(u) = z$ for $Pr(X < z) < u \leq c$. For any event E of probability at least c ,

$$\begin{aligned} E[X 1_E] - J_c &= E[(X - z)(1_E - 1_B)] + z[Pr(E) - c] \\ &\geq 0. \end{aligned}$$

The first integrand is nonnegative, and $z \geq 0$. When the two events have equal probability, equality requires that integrand to vanish almost surely, giving the stated characterization. Conversely, any such event makes the integrand vanish, so equality holds. For $c = 0$ take the empty event; for $c = 1$ take the whole space. $\square$

Apply equation (B.1) to $X - \beta$ on the inactive event A^c . The lower quantile of $X - \beta$ is $q - \beta$, so its retained integral is $J_c - c\beta$. Since $Y - \beta \geq 0$ on A , $Pr(A^c) \geq c$ and $X = Y$ on A^c ,

$$M_Y \geq \beta + E[(X - \beta) 1_{A^c}] \geq J_c + \alpha\beta.$$

Together with $M_Y + M_S = M$ and $M_S \geq 0$, this proves the bounds in equation (9). To attain every pair, take $X = q(U)$ with U uniform on $(0, 1)$ and, for $0 \leq t \leq 1$, set

$$S_t = t(X - \beta)1_{(U > c)}, Y_t = X - S_t.$$

These variables satisfy all the constraints and $E[S_t] = tK_\alpha$. Thus the complete joint mean-energy set is the segment specified by equation (9). If $K_\alpha = 0$, that segment is a singleton. The construction includes $\alpha = 0, 1$, constant total energies, and zero total energy.

If instead $\beta = 0$, $X > 0$ almost surely and $Y > 0$ is required, the lower background endpoint is excluded when $\alpha > 0$. Indeed, if $Pr(A) > 0$, then

$$M_Y = E[X 1_{A^c}] + E[Y 1_A] > J_c.$$

If $Pr(A) = 0$, then $M_Y = M > J_c$, because the upper α fraction has positive energy. Taking $t < 1$ in the construction attains every remaining value and approaches the excluded endpoint. For $\alpha = 0$, the background mean is M .

The common-part decomposition for ordered couplings is given by Nutz and Wang [22], Proposition 2.6; its maximal equality property is discussed by Chen, Lin and Wang [23], Sec. 2. The next theorem states its implication for the activity constraint. Write ν_X and ν_Y for the two marginal probability measures, and use $d_{TV}(\nu_X, \nu_Y) = \sup_E |\nu_X(E) - \nu_Y(E)|$.

Theorem B.2 (Compatible Background-Law). A proposed background law ν_Y supported on $[\beta, \infty)$ is compatible with the total law ν_X , meaning that there is a joint law of (Y, S) with $Y + S \sim \nu_X$, $Y \sim \nu_Y$, $S \geq 0$ and $Pr(S > 0) \leq \alpha$, if and only if $F_Y(x) \geq F_X(x)$ for every x and $d_{TV}(\nu_X, \nu_Y) \leq \alpha$. Under the stochastic-order condition, the minimum of $Pr(X > Y)$ over couplings with $Y \leq X$ is exactly $d_{TV}(\nu_X, \nu_Y)$, and is attained.

Proof. An ordered coupling gives $F_Y \geq F_X$. For every Borel set E ,

$$|\nu_X(E) - \nu_Y(E)| \leq E |1_{(X \in E)} - 1_{(Y \in E)}| \leq Pr(X > Y).$$

This proves necessity and the lower bound on activity.

For sufficiency, let $\nu = \nu_X + \nu_Y$, with densities $f_X = d\nu_X/d\nu$ and $f_Y = d\nu_Y/d\nu$. Define the common subprobability measure by $d\nu_\cap = \min(f_X, f_Y)d\nu$. The positive and negative parts of $f_X - f_Y$ each integrate to $d_{TV}(\nu_X, \nu_Y)$, so $\nu_\cap$ has mass $1 - d_{TV}(\nu_X, \nu_Y)$. If $d_{TV}(\nu_X, \nu_Y) = 0$, the identical coupling suffices. For $d_{TV}(\nu_X, \nu_Y) > 0$, define residual probability measures

$$\nu_X^r = \frac{\nu_X - \nu_\cap}{d_{TV}(\nu_X, \nu_Y)}, \nu_Y^r = \frac{\nu_Y - \nu_\cap}{d_{TV}(\nu_X, \nu_Y)}.$$

Their distribution functions satisfy $F_{\nu_Y^r} - F_{\nu_X^r} = (F_Y - F_X)/d_{TV}(\nu_X, \nu_Y) \geq 0$. Their quantiles, coupled at the same uniform rank, therefore give $Y \leq X$. These residual measures are mutually singular. Any diagonal part

of their coupling would define a measure dominated by both residual marginals, so that part has mass zero. Hence $Y < X$ almost surely on the residual branch.

Couple the common measure identically with probability $1 - d_{TV}(v_X, v_Y)$, and use the residual coupling with probability $d_{TV}(v_X, v_Y)$; omit the common branch when $d_{TV}(v_X, v_Y) = 1$. The resulting marginals are v_X, v_Y , the coupling is ordered, and its activity probability is $d_{TV}(v_X, v_Y)$. $\square$

Contamination models instead formulate approximate stochastic order through trimming [38], Sec. 2. Theorem B.2 retains exact stochastic order and bounds the probability of a positive source contribution; the two criteria are distinct.

This criterion concerns existence of a compatible pairing. It does not verify an already prescribed temporal pairing. For any additional constraints on the total law, the attribution set is the union of the segments in equation (9) over the total laws satisfying those constraints. Both inclusions follow from the bound and construction for each fixed total law.

B.2. Exact percentile reports and a fixed total mean

Fix the exact package and support bounds of Appendix A.2, with $k \geq 0$ and $0 \leq \beta \leq a \leq b < \infty$. Give l and h their left-continuous representatives, as in that appendix; in particular, $h(u_j) = q_j$. For a compatible quantile

q , write $J_c(q) = \int_0^c q(u) du$. No constraint other than the exact anchors, support bounds and stated mean is imposed on the total law.

Proposition B.3 (Retained energy at a fixed mean). Let M be a feasible mean for the exact package, in the sense of Theorem 3.1. Among quantiles satisfying these anchors and support bounds,

$$\inf_{q: \int_0^1 q = M} J_c(q) = \underline{E}_c^{(M)} = \max \{ \underline{E}_c, M - U_{tail} \}.$$

The infimum is attained if and only if

$$M > \underline{E}_c + U_{tail} \text{ or } q_j = a \text{ for every } j \text{ such that } u_j \leq c. \quad (\text{B.2})$$

Proof. Proposition A.2 at rank one gives the feasible mean interval $(\underline{E}_-, E_+]$, with its lower endpoint included exactly when every reported anchor equals a . The condition is vacuous for $k = 0$. Every compatible quantile satisfies $l \leq q \leq h$, apart from immaterial endpoint values. Consequently,

$$J_c(q) \geq \underline{E}_c, J_c(q) = M - \int_c^1 q(u) du \geq M - U_{tail}.$$

These inequalities establish the asserted lower bound.

If $c = 0$, the retained integral is zero and the anchor condition in equation (B.2) is vacuous. If $c = 1$, it equals M , and equation (B.2) holds by mean feasibility. If $a = b$, the total law is constant and the conclusion follows directly. Assume henceforth $0 < c < 1$ and $a < b$.

Construction before imposing the exact anchors. There is a nondecreasing, left-continuous $\tilde{q}$ between l and h with total integral M and retained integral $\underline{E}_c^{(M)}$. To construct it when $M \leq \underline{E}_c + U_{tail}$, set

$$\tilde{q}(u) = \begin{cases} l(u), & 0 < u \leq c, \\ (1-t)l(u) + th(u), & c < u < 1, \end{cases}$$

where $t \in [0, 1]$ is chosen to satisfy

$$M = \underline{E}_- + t[\underline{E}_c + U_{tail} - \underline{E}_-].$$

Feasibility and the assumed case place M between the two endpoints of this affine expression. If its coefficient is zero, take $t=0$. The function cannot decrease at c , since its right-hand values are at least the corresponding values of l .

When $M > \underline{E}_c + U_{tail}$, instead set

$$\tilde{q}(u) = \begin{cases} (1-t)l(u) + th(u), & 0 < u \leq c, \\ h(u), & c < u < 1, \end{cases}$$

and choose $t \in (0, 1]$ so that

$$M = \underline{E}_c + U_{tail} + t[E_+ - \underline{E}_c - U_{tail}].$$

Here the coefficient is positive. Values on the left of c are bounded above by $h(c)$, and those on the right are no smaller. Thus this function is also nondecreasing. Each construction uses the left-hand value at c and is left-continuous. The specified integrals follow by integration. Exact anchor values have not yet been imposed.

Exact-anchor approximation with the mean preserved. Let q_ϵ be the exact-anchor quantile constructed in Appendix A.2; for $k=0$, set $q_\epsilon = l = a$. Define

$$\tilde{q}_\epsilon = \max\{\tilde{q}, q_\epsilon\}, M_\epsilon = \int_0^1 \tilde{q}_\epsilon(u) du.$$

The maximum is nondecreasing, left-continuous and supported on $[a, b]$. At every reported rank, $\tilde{q}(u_j) \leq h(u_j) = q_j$ and $q_\epsilon(u_j) = q_j$, so its anchors are exact. Moreover,

$$0 \leq M_\epsilon - M = \int_0^1 (\tilde{q}_\epsilon - \tilde{q}) \leq \int_0^1 (q_\epsilon - l) \leq k\epsilon(b-a).$$

Suppose first that $\underline{E}_- < M < E_+$. By Proposition A.2, choose one exact-anchor quantile $q^{(0)}$ with mean $M_0 < M$. Set

$$t_\epsilon = \frac{M_\epsilon - M}{M_\epsilon - M_0}, q_\epsilon^{(M)} = (1-t_\epsilon)\tilde{q}_\epsilon + t_\epsilon q^{(0)}. \quad (\text{B.3})$$

The denominator is positive and $0 \leq t_\varepsilon < 1$. Pointwise interpolation preserves support, monotonicity, left continuity and every exact anchor. Lemma A.1 realizes this function as the lower quantile of a compatible law. Equation (B.3) gives total integral exactly M . In addition,

$$0 \leq t_\varepsilon \leq \frac{k\varepsilon(b-a)}{M-M_0}, \int_0^1 |q_\varepsilon^{(M)} - \tilde{q}| \leq k\varepsilon(b-a) + t_\varepsilon(b-a) \rightarrow 0.$$

Hence the retained integrals converge to $\underline{E}_c^{(M)}$ while the reported mean and all anchors remain exact. These are quantile interpolations; no identity between interpolated quantiles and quantiles of mixture laws is used.

At $M = E_+$, the exact quantile h already has the required retained integral, since $E_+ - U_{tail} = \bar{E}_c$. If $M = E_-$ is feasible, every anchor equals a , $E_- = a$, and the constant quantile a attains the bound. This proves sharpness for every feasible mean.

Necessity of the attainment condition. If $M \leq \underline{E}_c + U_{tail}$, the bound is $\underline{E}_c$. Proposition A.2 shows that attaining this retained integral requires every anchor at or below c to equal a . In particular, an anchor at c is included because lower quantiles are left-continuous. This proves the necessary part of equation (B.2).

Sufficiency when the retained lower bound is active. Suppose $M \leq \underline{E}_c + U_{tail}$ and every anchor at or below c equals a . Then $l = a$ on $(0, c]$. Choose the intervals modified in constructing q_ε wholly above c for every anchor above c . The resulting q_ε equals a throughout the retained block, and its mean tends to E_- . The quantile $q^{(1)}$ equal to a on $(0, c]$ and h on $(c, 1)$ satisfies every anchor and has mean $\underline{E}_c + U_{tail}$.

For $M > E_-$, take ε small enough that $\int_0^1 q_\varepsilon < M$, and interpolate q_ε with $q^{(1)}$ to obtain mean M . The retained integral stays equal to $ac = \underline{E}_c$. If $M = \underline{E}_c + U_{tail}$, take $q^{(1)}$ alone. The feasible case $M = E_-$ has already been handled by the constant law.

Sufficiency when the tail bound is active. Suppose $M > \underline{E}_c + U_{tail}$. Define an exact-anchor quantile by using q_ε on $(0, c]$ and h on $(c, 1)$. There is no downward jump at c , because $q_\varepsilon \leq h$. Every anchor, including one at c , remains exact. Its tail integral is U_{tail} , and its retained integral tends to $\underline{E}_c$.

Choose ε so that this retained integral is below $M - U_{tail}$. The quantile h has the same tail and retained integral $\bar{E}_c \geq M - U_{tail}$. Interpolating the two quantiles therefore gives retained integral exactly $M - U_{tail}$ and total mean exactly M . This completes the proof of equation (B.2). $\square$

Proof of Theorem 4.1. For each compatible total law, Appendix B.1 gives background minimum $\alpha\beta + J_c(q)$ and source maximum $M - \alpha\beta - J_c(q)$. Proposition B.3 supplies their sharp endpoints and exact inclusion rule, proving equation (10). Zero source and background mean M are always attained. For any source mean strictly between zero and the sharp supremum, choose a compatible total law whose maximal source mean exceeds that value, and scale its source allocation as in Appendix B.1. This attains every strictly intermediate pair with sum M .

If $a > 0$, $\beta = 0$, $\alpha > 0$ and $Y > 0$ is required, Appendix B.1 gives $M_Y > J_c(q) \geq \underline{E}_c^{(M)}$ for every compatible law. The lower background endpoint is therefore excluded even when the retained-integral infimum is attained.

The same scaled constructions, with scale less than one, attain every interior value and approach that endpoint. $\square$

For the example in Section 4.2, the total law with masses $1/2, 1/4, 1/4$ at $1, 5, 9$ has median 1, mean 4, and $J_{3/4} = 7/4$. Assigning the energy 9 observations entirely to the source gives source mean $9/4$ and background mean $7/4$. Thus both refined endpoints are attained.

Without the total-mean constraint. Proposition A.2 gives the infimum $\alpha\beta + \underline{E}_c$ of background mean, with attainment exactly when every anchor at or below c equals a . The upper completion h , with zero source, attains background mean E_+ . Every intermediate background mean is attained by mixing two admissible joint laws that bracket it. Such a mixture preserves the support and activity bound. It also preserves each exact anchor: $F_X(q_j) \geq u_j$ and $F_X(x) < u_j$ for every $x < q_j$ remain true under convex combinations. On the single total law with quantile h , scaled allocations attain every source mean in $[0, U_{tail} - \alpha\beta]$. These facts prove the package-only assertions of Section 4.2.

For interval-valued reports, the union described in Section 2.4 retains the mean and every other supplied constraint in each exact-anchor class. The proof of Theorem 4.1 does not remove those constraints or establish independent substitution of interval limits.

B.3. Recovery from a separated inactive block

Proof of Theorem 4.2. Let $0 < \pi < 1$, $c_0 = 1 - \pi$, and choose a threshold strictly between the essential supremum of X on A^c and its essential infimum on A . Under separation, $q(c_0)$ equals the essential supremum of X on A^c , and A^c contains $\{X < q(c_0)\}$ and excludes $\{X > q(c_0)\}$ up to null sets. Since $Pr(A^c) = c_0$ and $X = Y$ on A^c , the equality case of Lemma B.1 gives

$$c_0 M_{off} = E[X 1_{A^c}] = J_{c_0}.$$

Decomposition over activity and inactivity yields

$$M_Y = c_0 M_{off} + \pi E[Y | A] = M_{off} + \pi\delta.$$

Because $\pi > 0$, the whole-window background mean equals M_{off} if and only if $\delta = 0$. This proves equation (11) and its mean-recovery condition.

If Y is independent of 1_A , its conditional law on inactivity is its marginal law. Below the separating threshold, therefore, $F_X(x) = c_0 F_Y(x)$. For $0 < u < 1$, put $z = q_Y(u)$. This value lies below that threshold because the law of Y equals its conditional law on A^c , whose essential supremum is below the threshold. The exact-anchor criterion of Section 2.1, applied to Y and scaled by c_0 , gives $F_X(z) \geq c_0 u$ and $F_X(x) < c_0 u$ for every $x < z$. Hence $q(c_0 u) = q_Y(u)$, including ranks at atoms.

Call a maximal nonempty open interval on which F_X is constant a plateau, and call the constant value its height. A plateau is nontrivial if its height lies strictly between zero and one. Suppose additionally that the support of Y is connected. Every nonempty open interval between two support points then has positive background probability. Consequently, the inactive block has no internal open interval on which F_X is

constant. Separation creates a plateau of height c_0 after this block. No plateau of positive height occurs before it. For singleton background support, the initial atom is followed by the separating plateau, with the same conclusion. Thus the height of the first nontrivial plateau identifies c_0 within the stated model.

If operating information directly identifies A^c , its conditional mean is observable without separation. The decomposition $M_Y = M_{off} + \pi\delta$ still governs extension to the whole window. An upper bound α identifies neither A^c nor π ; when $\pi < \alpha$, the retained fraction $1 - \alpha$ is smaller than c_0 .

The identity $q_Y(u) = q(c_0 u)$ also specifies the finite-report limitation. An anchor at total rank $c_0 u_j$ becomes a background anchor at rank u_j . With declared background support bounds, Proposition A.2 gives the corresponding mean interval. Sharpness under these constraints follows by embedding each bounded background completion in the separated independent model: choose $z > \text{ess sup } Y$, take A independently of Y , and set $S = (z - Y)1_A$. Then $X = Y$ off activity and $X = z$ on activity. Sharpness under additional total-mean or active-block constraints requires a construction satisfying those constraints as well. A finite package alone generally leaves the retained integral undetermined. $\square$

B.4. Recovery and selection error

Proof of Theorem 4.3. Assume $0 < \pi < 1$ and set $c_0 = 1 - \pi$. Let B select the lowest mass c_0 , as in Lemma B.1, on a probability-space extension preserving the original joint law. Since $\Pr(A^c) = c_0$, equation (B.1) gives

$$D_{sel} = E[X 1_{A^c}] - J_{c_0} \geq 0.$$

Using $X = Y$ on A^c and the conditional-mean decomposition from Appendix B.3,

$$\frac{J_{c_0}}{c_0} - M_Y = M_{off} - \frac{D_{sel}}{c_0} - (M_{off} + \pi\delta) = -\pi\delta - \frac{D_{sel}}{c_0}.$$

This is equation (12), without any separation or independence assumption.

For every admissible choice of B , the sets $A^c \setminus B$ and $B \setminus A^c$ have equal probability because $\Pr(A^c) = \Pr(B) = c_0$. If $a \leq X \leq b$, cancelling their common intersection in the two integrals gives

$$\begin{aligned} D_{sel} &= E[X 1_{A^c \setminus B}] - E[X 1_{B \setminus A^c}] \\ &\leq (b - a) \Pr(A^c \setminus B). \end{aligned}$$

Combining this inequality with $D_{sel} \geq 0$ and $-\pi|\delta| \leq -\pi\delta \leq \pi|\delta|$ proves the error bounds in Theorem 4.3.

Lemma B.1 also shows that $D_{sel} = 0$ precisely when A^c is a minimum-energy selection, allowing arbitrary choices within the threshold atom. Separation is sufficient for this property. If $\delta = 0$ but $D_{sel} > 0$, the recovery error is strictly negative. In general, δ can have either sign and the two terms can cancel. For example, assign probability $1/2$ to each of $(Y, S) = (2, 0)$ and $(Y, S) = (1, 1/2)$. Then $c_0 = 1/2$, $M_Y = 3/2$, $J_{c_0} = 3/4$, $\delta = -1$ and $D_{sel} = 1/4$. The recovery error is zero although the active total energy is below the

inactive total energy. Equality of the retained-block mean and the actual background mean therefore does not imply separation. $\square$

B.5. Percentile consequences and attribution error

For every finite x , the relations $Y \leq X$ and $X = Y$ on A^c imply

$$F_X(x) \leq F_Y(x) \leq F_X(x) + \alpha.$$

Fix $\alpha < u < 1$ and set $z = q_Y(u)$. Since $F_Y(z) \geq u$, the upper CDF inequality gives $F_X(z) \geq u - \alpha$, hence $q(u - \alpha) \leq z$. The lower CDF inequality gives $z \leq q(u)$. This proves the background-percentile bracket in Section 4.4 without a continuity assumption.

Together with Appendices B.1 to B.4, this establishes the three information regimes of Section 4.4. Exact recovery uses the identified separated block and $\delta = 0$; an activity bound alone gives the attribution intervals and percentile bracket. With no informative activity restriction, set $\alpha = 1$ in equation (9) to obtain $I_Y = [\beta, M]$. Requiring activity probability exactly one would define a different class.

Sparse activity. For $0 < \alpha < 1/2$, nonnegativity and quantile monotonicity give

$$J_{1-\alpha} \geq \int_{\alpha}^{1-\alpha} q(u) du \geq (1-2\alpha)q(\alpha).$$

If $q(\alpha) > 0$, left continuity also gives $\int_0^{\alpha} q(u) du > 0$, so the displayed bound on $J_{1-\alpha}$ is strict. Equation (9) then proves the strict energy inequality in equation (13); taking logarithms gives its strict level bound. The sharp constant is therefore an unattained supremum. At $\alpha = 0.05$, $q(\alpha)$ corresponds to L_{95} and the constant is $10 \log_{10}(10/9)$.

The constant is sharp over the nonnegative class with $\beta = 0$. For $0 < \vartheta < \alpha$, let $X = \vartheta$ with probability $\alpha - \vartheta$ and $X = 1$ otherwise. Then $q(\alpha) = 1$ and

$$J_{1-\alpha} = (\alpha - \vartheta)\vartheta + 1 - 2\alpha + \vartheta \rightarrow 1 - 2\alpha.$$

The extremizing allocation of Appendix B.1 has $M_Y = J_{1-\alpha}$, so its logarithmic error approaches the stated constant. In the opposite direction, let $X = 1$ with probability α and $X = z > 1$ otherwise. Then $q(\alpha) = 1$ and the same allocation gives

$$M_Y = J_{1-\alpha} = \alpha + (1 - 2\alpha)z \rightarrow \infty.$$

Thus underestimation is unbounded when no common upper support bound is imposed. Every member of both constructions has finite mean.

Exact inactivity during at least 5%. With $\beta = 0$, this assumption corresponds to $\alpha = 0.95$, so equation (9) gives $J_{0.05} \leq M_Y \leq M$. When $q(0.05) > 0$, left continuity gives $J_{0.05} > 0$, and monotonicity gives $J_{0.05} \leq 0.05 q(0.05)$. The lower endpoint is attained by the allocation in Appendix B.1; hence its level lies

at least $10 \log_{10} 20$ dB below the total-record L_{95} . This conclusion uses $\beta=0$: for a positive floor the attained minimum is $J_{0.05} + 0.95\beta$. If Theorem 4.2 also applies, $\delta=0$ and $c_0 > 0.05$, then

$$M_Y = \frac{J_{c_0}}{c_0} \geq \frac{1}{c_0} \int_{0.05}^{c_0} q(u) du \geq \frac{c_0 - 0.05}{c_0} q(0.05).$$

For positive $q(0.05)$, this proves the logarithmic comparison in Section 4.5. At $c_0=0.05$, monotonicity instead gives $M_Y \leq q(0.05)$. Equality holds if and only if $q(u)=q(0.05)$ for almost every $0 < u < c_0$, equivalently if the inactive energy is almost surely constant.

The limits of these statements follow from explicit separated independent constructions. In each, choose A independently of Y , with $\Pr(A^c) = c_0$, and set $S = (z - Y)1_A$ for a constant $z > \text{ess sup } Y$.

For $c_0=0.05$, take $Y=\vartheta$ with probability $1-\vartheta$ and $Y=1$ with probability ϑ , where $0 < \vartheta < 1$, and set $z=2$. Then $q(0.05)=1$ but $M_Y = 2\vartheta - \vartheta^2 \rightarrow 0$. The logarithmic overestimation is unbounded. For any fixed $c_0 > 0.05$, take $Y=1$ with probability $0.05/c_0$ and $Y=x > 1$ otherwise, with $z=x+1$. Now $q(0.05)=1$ and

$$M_Y = \frac{0.05}{c_0} + \left(1 - \frac{0.05}{c_0}\right)x \rightarrow \infty.$$

These laws establish unbounded underestimation at every such fixed cutoff and both unbounded directions when c_0 is unknown in $[0.05, 1)$. All energies are positive, separation holds, and background is independent of activity.

Quantified insignificant contribution. Replace the exact activity constraint by $\Pr(S > \rho Y) \leq \alpha$, where $\rho \geq 0$ and $\beta=0$. On $E = S \leq \rho Y$, one has $Y \geq X/(1+\rho)$, and $\Pr(E) \geq c$. Equation (B.1) therefore yields

$$M_Y \geq \frac{E[X 1_E]}{1+\rho} \geq \frac{J_c}{1+\rho}.$$

Exact inactivity is the case $\rho=0$. A qualitative significance condition alone specifies no value of ρ .

Source attribution. Since $M_S = M - M_Y$, substituting $q(0.05)$ for background mean gives source error $M_Y - q(0.05)$. Under $\alpha=0.05$, equation (13) implies $0 \leq q(0.05) \leq M_Y/0.90$. Hence

$$-\frac{M_Y}{9} \leq M_Y - q(0.05) \leq M_Y,$$

including zero energy. The left inequality is strict when $q(0.05) > 0$, as in equation (13); the non-strict enclosure also covers zero energy.

All four cases of Table 2 occur under exact inactivity during at least 5%, even with separation and independent background. Use the preceding independent construction with the following choices.

1. For $c_0=1/2$, let Y have equal probabilities at 1 and 3, and take $z=4$. Then $q(0.05)=1 < M_Y=2$, giving overestimated source energy.
2. For $c_0=1/2$, let $Y=1$ and $z=2$. Then $q(0.05)=M_Y=1$, giving exact source energy.

3. For $c_0 = 1/20$, let $Y = 1/100$ with probability $99/100$ and $Y = 1$ with probability $1/100$, and take $z = 2$. Then $M_Y = 199/10000 < q(0.05) = 1 < M = 380199/200000$, giving underestimated but nonnegative source energy.
4. Retain the background law and c_0 of the third construction, but take $z = 101/100$. Then $q(0.05) = 1 > M = 192099/200000$, giving a negative source subtraction.

B.6. Indivisible observation cells

Let $n_{rec} \geq 1$ be the number of fixed cells, with finite observed energies $x_i \geq \beta$ and positive durations T_i . Write $w_i = T_i / \sum_j T_j$, so $\sum_i w_i = 1$. A cell receives one background allocation $y_i \in [\beta, x_i]$ and source allocation $s_i = x_i - y_i$. The activity constraint is $\sum_{i: s_i > 0} w_i \leq \alpha$. These weights describe the stipulated observation basis. Subsets of cell indices are regarded as events of the finite probability space with masses w_i .

Proposition B.4 (Finite-Record Selection). Set $M = \sum_i w_i x_i$ and

$$K_{rec} = \max_{B \subseteq \{1, \dots, n_{rec}\} : \sum_{i \in B} w_i \leq \alpha} \sum_{i \in B} w_i (x_i - \beta). \quad (\text{B.4})$$

The attainable mean-energy pairs are exactly $M_Y + M_S = M$ with $0 \leq M_S \leq K_{rec}$. For equal weights, put $k_{act} = \lfloor n_{rec} \alpha \rfloor$ and order the energies as $x_{(1)} \leq \dots \leq x_{(n_{rec})}$. Then

$$\min M_Y = \frac{1}{n_{rec}} \left(\sum_{i=1}^{n_{rec}-k_{act}} x_{(i)} + k_{act} \beta \right).$$

If $\beta = 0$, every $x_i > 0$ and every $y_i > 0$ is required, the lower background endpoint and upper source endpoint are excluded whenever $K_{rec} > 0$. If $K_{rec} = 0$, only $(M, 0)$ is attainable.

Proof. For any allocation, its active index set is feasible in equation (B.4), and

$$M_S = \sum_{i: s_i > 0} w_i s_i \leq \sum_{i: s_i > 0} w_i (x_i - \beta) \leq K_{rec}.$$

There are finitely many subsets, so a maximizing subset exists. On that subset set $s_i = t(x_i - \beta)$, and set $s_i = 0$ elsewhere, for $0 \leq t \leq 1$. Its active weight does not exceed the subset's weight, and its source mean is $t K_{rec}$. This proves both endpoints and every intermediate pair.

For equal weights, feasible subsets have at most k_{act} elements. Because all $x_i - \beta$ are nonnegative, one may select k_{act} elements. Exchanging a selected smaller energy for an unselected larger one cannot decrease the source sum. The largest k_{act} energies therefore maximize it, yielding the stated background minimum.

Under strict positivity, every nonempty active set satisfies

$$M_S < \sum_{i:s_i>0} w_i x_i \leq K_{rec}.$$

When $K_{rec} > 0$, its upper endpoint is consequently impossible. The maximizing-subset construction with $t < 1$ keeps every background allocation positive and attains every smaller nonnegative source mean. If $K_{rec} = 0$, no positive source mean is feasible. $\square$

Applied to the discrete total law with masses w_i , equation (9) permits subdivision of a cell's probability mass. Every indivisible allocation is admissible in that larger class, so $K_{rec} \leq K_\alpha$. Equality need not hold: with a single positive-energy cell, $\beta = 0$ and $0 < \alpha < 1$, no whole cell may be active, whereas fractional allocation permits positive source energy. For unequal durations, equation (B.4) retains the required subset constraint.

Appendix C. Proofs for Section 5: percentile arithmetic and pooling

All quantiles are lower quantiles unless a superscript $+$ is shown, and independence means mutual independence. Component laws are Borel probability laws. The generalized-inverse representation of Appendix A applies at atoms. Sections C.1 to C.2 prove the core percentile and mixed-descriptor results, Section C.3 supplies the interpolation argument they use, and Section C.4 proves pooling. The complete activity, product-root, finite-reduction and dependence proofs are in Supplement Sections S.5 to S.8; their result statements remain in Section 5 of the article.

C.1. Exact percentile bounds

Proof of Proposition 5.1. For any coupling of nonnegative components, $X \geq X_i$ implies $q_X(p) \geq q_{X_i}(p)$. Substitution into the definition of $D_p^{(m)}$ proves Proposition 5.1 and equation (14). The sharp fixed-mean refinement (15) is proved below. $\square$

Proof of Theorem 5.2. We prove the bound with the upper quantile in its conclusion. Put $\lambda_i^{anc} = \Pr(X_i < \eta_i) \leq p$ and $B_i' = 1_{\{X_i \geq \eta_i\}}$. For $x < e_p$, define the downward-closed Boolean set

$$A_x = \left\{ b \in \{0, 1\}^m : \sum_i \eta_i b_i \leq x \right\}.$$

The independent indicators B_i' have zero probabilities $\lambda_i^{anc} \leq p$. Increasing these probabilities to p cannot decrease the probability of a downward-closed event. Therefore

$$F_X(x) \leq \Pr(B' \in A_x) \leq \Pr(B \in A_x) = F_W(x) \leq p. \quad (C.1)$$

If $F_W(x) < p$, equation (C.1) already gives the required strict inequality. Suppose $F_W(x) = p$; it remains to exclude equality throughout that equation. Call a coordinate essential if changing it changes membership in A_x for some configuration of the other coordinates. For any essential coordinate, increase its zero probability last, after setting every other zero probability to p . Every configuration of those other coordinates then has positive probability. Each one-coordinate increase is nondecreasing, so equality of the initial and final probabilities forces every increment to vanish. If $\lambda_i^{anc} < p$, this last increase is strict. Thus equality in the second inequality of (C.1) requires $\lambda_i^{anc} = p$ for every essential coordinate.

Since $F_W(x) = p \in (0, 1)$, the set has a boundary edge: for some essential i , some $b \in A_x$ has $b_i = 0$ and $b + e_i \notin A_x$, where e_i is the i th coordinate vector. Nonessential coordinates do not affect either membership statement; fix each at a status of positive probability. All essential statuses also have positive probability because their zero probabilities equal p .

Exact anchoring and $\lambda_i^{anc} = p$ give, for every $y < \eta_i$,

$$\Pr(y < X_i < \eta_i) = p - F_{X_i}(y) > 0.$$

Apply this with $y = x - \sum_j \eta_j b_j < \eta_i$. Independence supplies positive probability that the status vector is b but $X > x$: component i exceeds y , and the components with status one contribute at least their anchors. This makes the first inequality in (C.1) strict. Hence $F_X(x) < p$ for every $x < e_p$, proving the lower bound.

For attainment, independently give component i mass $p - \varepsilon$ at zero and $1 - p + \varepsilon$ at η_i , with $0 < \varepsilon < p$. Its lower p -quantile is exactly η_i . The probability that the sum is at most e_p tends, as $\varepsilon \downarrow 0$, to $F_W(e_p) > p$. The strict inequality follows because W has finite support and e_p is its upper p -quantile. For sufficiently small ε , the constructed sum therefore has lower quantile at most e_p . The bound just proved makes it equal to e_p , establishing both extrema in equation (16). $\square$

Computing the bound. Set $p = 1 - N / 100$ and convert each reported component L_N to its energy anchor η_i . Enumerate the subset sums of the anchors, weighting each subset by the product of $1 - p$ over included components and p over excluded components. Combine the probabilities of equal sums and sort the distinct sums. The first cumulative probability strictly greater than p selects e_p . Report $10 \log_{10} e_p$ as the sharp lower bound on the combined level and use equation (16) for the maximum overestimation. The Bernoulli law computes a bound; it is not a fitted source model. A lower-quantile routine gives a valid but nonsharp bound wherever the two inverses of W differ at p .

For two sources, suppose $\eta_1 < \eta_2$. Then $F_W(\eta_1) = p$ and $F_W(\eta_2) = 2p - p^2 > p$, so $e_p = \eta_2$. If the anchors coincide, $F_W(0) = p^2 < p$ and $F_W(\eta_1) = 2p - p^2 > p$. Interchanging the indices covers the remaining order. Thus the independent and arbitrary-dependence lower endpoints coincide for two sources.

For equal anchors, $W = \eta_1 N_{bin}$. At $p = 1 / 10$, direct expansion of the binomial probabilities gives

$$(\kappa_{2,p}, \kappa_{3,p}, \kappa_{4,p}, \kappa_{5,p}, \kappa_{6,p}) = (1, 2, 3, 4, 4).$$

The cumulative probabilities immediately before and at these crossings are, respectively,

$$(0.01, 0.19), (0.028, 0.271), (0.0523, 0.3439), \\ (0.08146, 0.40951), (0.01585, 0.114265).$$

These finite decimals are exact. Equation (16) gives the stated decibel constants and their increase from five to six sources. For anchors $(1, 100, 100)$, $F_W(100) = 0.028$ and $F_W(101) = 0.19$, giving $e_p = 101$ and $10 \log_{10}(201 / 101) = 2.9887$ dB to four decimal places.

For fixed p , the variance of N_{bin} / m is $p(1 - p) / m$. Chebyshev's inequality puts its lower-tail probability below p at every fixed distance below $1 - p$, and above p at every fixed distance above $1 - p$, for all sufficiently large m . Consequently $\kappa_{m,p} / m \rightarrow 1 - p$. Continuity of the logarithm proves the limit stated in Section 5.2.

For the absence of a finite upper endpoint, give each independent component mass p at its positive anchor and $1 - p$ at a common $s > \sum_i \eta_i$. The all-low probability is $p^m < p$; every other sum is at least s . Thus $q_X(p) \geq s \rightarrow \infty$, preserving every anchor. For two unit anchors the same computation applies for every

$s > 1$: $F_X(2) = p^2 < p$ and $F_X(s+1) = 2p - p^2 > p$, giving an explicit example of unbounded underestimation.

Proof of the exact identification sets. For any $z \geq \max_i \eta_i$, choose finite means satisfying $\mu_i > (1-p)\eta_i$ and $\sum_i \mu_i > (1-p)z$; the construction proving equation (15) below attains z with the exact anchors.

Under independence, interpolate by Lemma C.1 (Appendix C.3) between the bounded attaining laws of Theorem 5.2 and the preceding unbounded-endpoint witnesses with $s > \sum_i \eta_i$. Both systems have the same exact anchors and lie in $[0, s]$; their combined quantiles are e_p and at least s . Interpolation attains every value in $[e_p, s]$. Since s is arbitrary, the exact sets are $[e_p, \infty)$ under independence and $[\max_i \eta_i, \infty)$ under arbitrary dependence. $\square$

Proof of equation (15). Appendix A.2 gives the strict feasibility conditions $\mu_i > (1-p)\eta_i$. Necessity of the lower bound follows from $X \geq X_i$. Since $q_X(p) > 0$, the same single-anchor argument gives $\sum_i \mu_i > (1-p)q_X(p)$, excluding the upper endpoint.

Fix any z between the asserted endpoints. We construct a finite joint law with total quantile z . For each $j = 1, \dots, m$, choose a vector $\mathbf{t}^{(j)}$ of nonnegative component energies with sum z and $t_j^{(j)} = \eta_j$. If $z \leq \sum_i \eta_i$, choose every $t_i^{(j)} \leq \eta_i$; this is possible because $0 \leq z - \eta_j \leq \sum_{i \neq j} \eta_i$. If $z > \sum_i \eta_i$, start with every component at its anchor and assign the excess to the component following j cyclically; $m \geq 2$ ensures that this leaves component j at its anchor.

Assign probability ε to each vector, on disjoint events E_j . On a further event L , set all components to zero, taking

$$Pr(L) = \begin{cases} p - m\varepsilon + \varepsilon/2, & z \leq \sum_i \eta_i, \\ p - \varepsilon/2, & z > \sum_i \eta_i. \end{cases}$$

On the remaining event set component i equal to

$$H_i^\varepsilon = \frac{\mu_i - \varepsilon \sum_{j=1}^m t_i^{(j)}}{1 - Pr(L) - m\varepsilon}.$$

For sufficiently small $\varepsilon > 0$, all probabilities and energies are nonnegative. Moreover, $H_i^\varepsilon \rightarrow \mu_i / (1-p) > \eta_i$ and $\sum_i H_i^\varepsilon \rightarrow \sum_i \mu_i / (1-p) > z$. Thus the remaining event lies above every anchor and above total energy z . In the first case, the probability below anchor η_i is at most $p - \varepsilon/2$, and the probability at or below it is $p + \varepsilon/2$. In the second case, all event vectors lie at or above every anchor, and the event E_i raises the cumulative probability at η_i from $p - \varepsilon/2$ to at least $p + \varepsilon/2$. Every anchor is therefore exact. The means

hold by construction. Finally, $\Pr(X < z) = \Pr(L) < p$, whereas $\Pr(X \leq z) = \Pr(L) + m\varepsilon \geq p$. This attains every z in (15), including its lower endpoint. $\square$

Proof of the remaining elementary sets in Table 3. With only positive means reported, put $M = \sum_i \mu_i$. For any $0 < z < M / (1 - p)$, Appendix A.2 supplies a bounded nonnegative variable Z with mean M and lower p -quantile z . Set $X_i = (\mu_i / M) Z$. The components then have the prescribed means and their sum is Z . For $z = 0$, give Z mass p at zero and $1 - p$ at $M / (1 - p)$; the same allocation applies. The strict single-anchor inequality excludes the upper endpoint. With only anchors reported, the component finite means range over $((1 - p)\eta_i, \infty)$ by Appendix A.2. Their sums therefore range over $((1 - p)\sum_i \eta_i, \infty)$, also under independence. $\square$

Proof of the percentile-subtraction assertions. For two components, the exact combined-percentile set is $[\max(\eta_Y, \eta_S), \infty)$ under either dependence assumption. For fixed $x \geq \eta_Y > 0$, compatibility is therefore equivalent to $0 < \eta_S \leq x$, and every such anchor is attained. If $x > \eta_Y$, subtraction overestimates by $10 \log_{10}[(x - \eta_Y) / \eta_S]$, which diverges as $\eta_S \downarrow 0$. Its greatest underestimation is $10 \log_{10}[x / (x - \eta_Y)]$, attained at $\eta_S = x$. If $x = \eta_Y$, the computed energy is zero for every admissible positive source anchor. The class here imposes no means, support ceilings or activity bounds. $\square$

Proof of the strictly positive infimum. Replace every zero atom of an attaining independent system by $\varepsilon_{pos} \in (0, \min_i \eta_i)$. All anchors remain exact. The coupled sums differ by at most $m\varepsilon_{pos}$, so equation (C.2) places the new combined quantile in $[e_p, e_p + m\varepsilon_{pos}]$. Letting $\varepsilon_{pos} \downarrow 0$ proves that e_p remains the sharp infimum. This argument makes no assertion of attainment in the strictly positive class. $\square$

Finally, in the two sign witnesses of Section 5.1, each component has probabilities $1/20, 1/20, 9/10$ at $1, 2, 8$ in both couplings. Each marginal therefore has mean $147/20$ and lower $1/10$ -quantile 2. The first coupling has cumulative probability $1/10$ at combined energy 3 and none below it. The second has cumulative probability $1/20$ at 2 and $3/20$ at 10. Their combined quantiles are therefore 3 and 10, their means both equal $147/10$, and their errors are $10 \log_{10}(4/3)$ and $10 \log_{10}(2/5)$. The two couplings thus have identical component marginals and total means but opposite signs of percentile-sum error.

C.2. Prescribed means and mixed-descriptor arithmetic

Proof of Proposition 5.3. Exact anchoring gives $\mu_i \geq (1 - p)\eta_i$. Equality would require mass p at zero and $1 - p$ at η_i : both the contribution below the anchor and the excess above it must vanish. That law has lower p -quantile zero. Hence feasibility requires the strict inequality $\mu_i > (1 - p)\eta_i$.

For sufficiency and attainment, choose independent preliminary variables $X_i^{(0)}$ with mass $p - \varepsilon$ at zero, mass ε uniformly distributed on $(0, \eta_i)$, and mass $1 - p$ at η_i . Their exact anchors are η_i , and their means are

$$\mu_i^{(0)} = (1 - p)\eta_i + \frac{\varepsilon \eta_i}{2}.$$

Choose $\varepsilon > 0$ sufficiently small that $\mu_i^{(0)} < \mu_i$ for every i and

$$g_{disc}(\varepsilon) = \sum_{b: \sum_i \eta_i b_i \leq e_p} (1-p)^{\sum_i b_i} (p-\varepsilon)^{m-\sum_i b_i} > p.$$

This is possible because $g_{disc}(0) = F_w(e_p) > p$. The expression counts only outcomes in which every preliminary variable takes a discrete value.

Independently replace $X_i^{(0)}$ by the constant

$$H_i^{rare} = \mu_i^{(0)} + \frac{\mu_i - \mu_i^{(0)}}{\phi_i^{rep}}$$

with probability $\phi_i^{rep} > 0$. Choose these probabilities small enough that $H_i^{rare} > \eta_i$, $\phi_i^{rep} < 1-p$, and $g_{disc}(\varepsilon) \prod_i (1-\phi_i^{rep}) > p$. The resulting component mean is exactly μ_i . Below η_i its distribution function is $(1-\phi_i^{rep})F_{X_i^{(0)}} < p$; at η_i it is $1-\phi_i^{rep} > p$. Thus every anchor remains exact. The event counted by the last product has combined energy at most e_p and probability exceeding p . Theorem 5.2 supplies the reverse quantile inequality. Each H_i^{rare} is finite, proving bounded attainment with no common prescribed cap. $\square$

For example, take three median-one components, $\varepsilon = \phi_i^{rep} = 1/100$, and $\mu_i = 10$. Then $\mu_i^{(0)} = 101/200$ and $H_i^{rare} = 190001/200$. The discrete preliminary event with sum at most two has probability $(99/100)^3 - 1/8 > 4/5$; the probability of no replacement exceeds $9/10$. Their intersection therefore has probability greater than $18/25 > 1/2$. All three means are exactly ten and the combined median remains two.

Proof of Proposition 5.4. For part 1, linearity gives $M_S = M - M_Y$. The background finite means compatible with the report and a nonnegative source are exactly $((1-p)\eta_Y, M]$. Every such mean is attained by a bounded background law and an independent constant source with the complementary mean. This proves compatibility and the asserted source-mean interval. For $M > \eta_Y$, positive source means tending to zero give unlimited overestimation by the corrected level; approaching the excluded upper endpoint gives the strict, sharp underestimation bound. If $(1-p)\eta_Y < M \leq \eta_Y$, the computed energy is nonpositive although positive source means remain possible.

For part 2, Appendix A.2 gives exactly $M_Y \in ((1-p)\eta_Y, \infty)$ for finite background means. Taking an independent constant source $S = M_S$ attains every asserted total mean. The lower bound on the total quantile follows from $X \geq Y$.

We attain every total quantile $z \geq \eta_Y$ by independent bounded laws. If $z = \eta_Y$, take $Y = \eta_Y$ and give S mass p at zero and $1-p$ at $M_S/(1-p)$. If $0 < z - \eta_Y < M_S/(1-p)$, keep $Y = \eta_Y$ and use Appendix A.2 to choose S with mean M_S and exact lower p -quantile $z - \eta_Y$. Translation gives the required total quantile. If $z - \eta_Y \geq M_S/(1-p)$, give Y mass p at η_Y and $1-p$ at a value greater than z . Independently, let $S = z - \eta_Y$ with probability $M_S/(z - \eta_Y) \leq 1-p$, and zero otherwise. Its mean is M_S , and

$$\Pr(X < z) = p \left(1 - \frac{M_S}{z - \eta_Y} \right) < p, \Pr(X \leq z) = p.$$

Thus every finite $z \geq \eta_Y$ is attained. Independent attainments also establish the arbitrary-dependence set. The mixed-sum error bounds follow by comparing Λ with the two exact sets. The mean bound is unattained, while the percentile bound is attained at $z = \eta_Y$. Their unbounded upper endpoints give unlimited underestimation. If $S = M_S$ almost surely, translation gives $q_X(p) = \eta_Y + M_S$. $\square$

C.3. Independent quantile interpolation

Lemma C.1 (Independent quantile interpolation). Consider two bounded independent systems satisfying the same reported constraints among means, support bounds, activity bounds and finitely many exact quantile anchors. Writing $q_i^{(0)}, q_i^{(1)}$ for their component lower quantiles, the interpolants

$$q_i^{(t)}(u) = (1-t)q_i^{(0)}(u) + tq_i^{(1)}(u), 0 \leq t \leq 1,$$

define feasible independent systems. For each $0 < r < 1$, their combined lower r -quantile is continuous in t . For nonnegative components, any common activity bounds are also preserved.

Proof. Each interpolant is nondecreasing and left-continuous, preserves every specified anchor, and satisfies each prescribed mean and any declared support bounds. Appendix A's converse makes it the lower quantile of $q_i^{(t)}(U_i^{rk})$, where the U_i^{rk} are mutually independent uniform ranks. If both endpoint curves lie in $[a_i, b_i]$, the sums at parameters t, t' differ almost surely by at most $|t - t'| \sum_i v_i$. For $\delta_{disp} \geq 0$, stochastic order gives the displacement inequality

$$|Z - Z'| \leq \delta_{disp} \text{ almost surely} \Rightarrow |q_Z(r) - q_{Z'}(r)| \leq \delta_{disp}. \quad (C.2)$$

Indeed $Z \leq Z' + \delta_{disp}$ and $Z' \leq Z + \delta_{disp}$ give the two quantile inequalities. Applying equation (C.2) proves Lipschitz continuity. An activity bound $\Pr(X_i > 0) \leq \alpha_i$ is equivalent, for nonnegative variables, to a zero quantile curve on the ranks $0 < u < 1$ satisfying $u \leq 1 - \alpha_i$. Interpolation preserves that property. $\square$

C.4. Pooling percentile reports across time

Proof of Proposition 5.10. Index the windows by i , with weights $w_i > 0$ summing to one, laws F_{X_i} , and anchors $\eta_i = q_{X_i}(p)$ at the common rank $0 < p < 1$. The pooled law is $F = \sum_i w_i F_{X_i}$. Write $\eta_{min} = \min_i \eta_i$ and $\eta_{max} = \max_i \eta_i$. By the definition of the lower quantile in equation (1), $F_{X_i}(\eta_i) \geq p$ and $F_{X_i}(z) < p$ for every $z < \eta_i$.

Each F_{X_i} is nondecreasing, so $F_{X_i}(\eta_{max}) \geq F_{X_i}(\eta_i) \geq p$ for every i , whence $F(\eta_{max}) \geq p$ and $q_X(p) \leq \eta_{max}$. For $z < \eta_{min}$ we have $z < \eta_i$ for every i , so $F_{X_i}(z) < p$ and therefore $F(z) < p$, whence $q_X(p) \geq \eta_{min}$. The pooled anchor therefore lies in $[\eta_{min}, \eta_{max}]$.

For attainment of the upper endpoint, let j attain the maximum. Give window j all its mass at η_j , and each other window mass p at η_i with the remainder at some $H > \eta_{\max}$. Then $F(z) \leq p \sum_{i \neq j} w_i < p$ for $z < \eta_{\max}$, while $F(\eta_{\max}) \geq w_j + p(1 - w_j) > p$.

Every remaining value is attained by one construction. Fix v with $\eta_{\min} \leq v < \eta_{\max}$ and let $A_v = \sum_{i: \eta_i \leq v} w_i$, so that $0 < A_v < 1$. Each window with $\eta_i \leq v$ takes mass p at η_i and the remainder at v . Each window with $\eta_i > v$ takes mass $p - \varepsilon$ at v , mass ε at η_i , and the remainder above $\eta_{\max}$, for a fixed ε with

$$0 < \varepsilon < \min \left\{ p, \frac{(1-p)A_v}{1-A_v} \right\}.$$

Every window keeps its anchor. In the first group, if $\eta_i < v$ then $F_{X_i}(z) = 0$ for $z < \eta_i$ and $F_{X_i}(\eta_i) = p$, while if $\eta_i = v$ the window is concentrated at v and its lower p -quantile is again η_i . In the second group $F_{X_i}(z) = 0$ for $z < v$ and $F_{X_i}(z) = p - \varepsilon < p$ for $v \leq z < \eta_i$, while $F_{X_i}(\eta_i) = p$.

For $z < v$ only the first group contributes and each of its windows contributes at most p , so $F(z) \leq p A_v < p$. At v the first group contributes A_v in full and the second contributes $p - \varepsilon$ each, so

$$F(v) = A_v + (1 - A_v)(p - \varepsilon) > p,$$

the inequality being exactly the choice of ε , since $A_v + (1 - A_v)(p - \varepsilon) > p$ holds if and only if $(1 - A_v)\varepsilon < (1 - p)A_v$. Hence $q_X(p) = v$. Taking $v = \eta_{\min}$ attains the lower endpoint.

The set is therefore exactly $[\eta_{\min}, \eta_{\max}]$. Applying $10 \log_{10}$ to positive anchors preserves the endpoints and their attainment. $\square$

Appendix D. Illustrative pooling witness

Proposition 5.10 states that the percentile of the pooled record can take every value between the smallest and the largest reported window percentile, with both endpoints attainable. Consider two equal-duration windows with $w_1 = w_2 = 1/2$, reporting L_{90} values of 40 dB and 50 dB. Their energy anchors are $\eta_1 = 10^4$ and $\eta_2 = 10^5$ at the common rank $p = 0.10$. The arithmetic average of their reported levels is 45 dB.

For the lower pooled endpoint, make each window constant at its own anchor. Each keeps its reported L_{90} , while the pooled distribution assigns mass $1/2$ to each anchor. Its lower 0.10-quantile is 10^4 , so its L_{90} is 40 dB.

For the upper endpoint, retain the second window at 50 dB. In the first window put mass $1/10$ at 40 dB and mass $9/10$ at 60 dB. Its lower 0.10-quantile remains 10^4 , preserving its reported L_{90} . The pooled law is now

$$Pr(X = 10^4) = 0.05, Pr(X = 10^5) = 0.50, Pr(X = 10^6) = 0.45.$$

Its cumulative probability below 10^5 is $0.05 < p$, and at 10^5 it is $0.55 \geq p$. Its lower 0.10-quantile is therefore 10^5 , so its L_{90} is 50 dB.

The two constructions preserve the same reported window percentiles and arithmetic average, yet the pooled L_{90} values are 40 dB and 50 dB. The average is 5 dB above one and 5 dB below the other. The unreported within-window mass determines the pooled rank; averaging the reported levels does not identify it.

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